# A CHARACTERIZATION OF DOMAINS IN $\mathbf{C}^2$ WITH NONCOMPACT AUTOMORPHISM GROUP

KAUSHAL VERMA

Abstract. Let $D$ be a bounded domain in $\mathbf{C}^2$ with a non-compact group of holomorphic automorphisms. Model domains for $D$ are obtained under the hypotheses that at least one orbit accumulates at a boundary point near which the boundary is smooth, real analytic and of finite type.

## 1. Introduction

Let $D$ be a bounded, or more generally a Kobayashi hyperbolic domain in $\mathbf{C}^n$. It is known that the group of holomorphic automorphisms of $D$, henceforth to be denoted by $\mathrm{Aut}(D)$, is a real analytic Lie group in the compact open topology of dimension at most $n^2 + 2n$; the maximal value occuring only when $D$ is biholomorphically equivalent to the unit ball $\mathbf{B}^n \subset \mathbf{C}^n$. This paper addresses the question of determining those domains $D$ for which $\mathrm{Aut}(D)$ is non-compact. By a theorem of H. Cartan, non-compactness of $\mathrm{Aut}(D)$ is equivalent to the existence of $p \in D$ and a sequence $\{\phi_j\} \in \mathrm{Aut}(D)$ such that $\{\phi_j(p)\}$ clusters only on $\partial D$. In other words, there is at least one point in $D$ whose orbit under the natural action of $\mathrm{Aut}(D)$ on $D$ accumulates at the boundary of $D$. Call $p_\infty \in \partial D$ an orbit accumulation point if it is a limit point for $\{\phi_j(p)\}$. In this situation it is known that local data regarding $\partial D$ near $p_\infty$ provides global information about $D$, the protoype example of this being the Wong-Rosay theorem. Indeed, it was shown in [46] that a smoothly bounded strongly pseudoconvex domain in $\mathbf{C}^n$ with non-compact automorphism group must be equivalent to $\mathbf{B}^n$. The same conclusion was arrived at in [39] under the weaker hypothesis that the boundary of $D$ is strongly pseudoconvex only near $p_\infty$. A systematic study of this phenomenon for more general pseudoconvex domains was initiated by Greene-Krantz and we refer the reader to [2], [3], [4], [5], [6], [7], [17], [20], [29] and [30] which provide a panoramic view of some of the known results in this direction. The survey articles [27] and [31] contain an overview of the techniques that are used and provide a comprehensive list of relevant references as well.

The main result in [46] was generalised for pseudoconvex domains in $\mathbf{C}^2$ (see [3]and [4] for related results) by Bedford-Pinchuk in [2]. They showed that a smoothly bounded weakly pseudoconvex domain in $\mathbf{C}^2$ with real analytic boundary for which $\mathrm{Aut}(D)$ is non-compact must be equivalent to the ellipsoid $E_{2m} = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$ for some integer $m \geq 1$. The corresponding local result (which would be the analogue of [39]) when the boundary of $D$ is smooth weakly pseudoconvex and of finite type only near $p_\infty$ was obtained in [6]. The model domain for $D$ is then not restricted to be $E_{2m}$ alone as above. It turns out that $D$ is equivalent to a domain of the form $G = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P(z_1, \overline{z}_1) < 0\}$ where $P(z_1, \overline{z}_1)$ is a homogeneous subharmonic polynomial of degree $2m$ (here $m \geq 1$ is an integer) without harmonic terms. Apart from translations along the imaginary $z_2$-axis, $G$ is invariant under the action of a one parameter subgroup of $\mathrm{Aut}(D)$ given by $s \mapsto S_s(z_1, z_2) = (\exp(s/2m) z_1, \exp(s) z_2)$ where $s \in \mathbf{R}$. As $s \to -\infty$, it can be seen that the orbit of any point $(z_1, z_2) \in G$ under the action of $(S_s)$ accumulates at the origin which lies on the boundary of $G$. The situation for domains in $\mathbf{C}^2$ was clarified further in [5]. It was shown that a smoothly bounded domain in $\mathbf{C}^2$ with real analytic boundary must be equivalent to $E_{2m}$. Thus pseudoconvexity of the domain that was assumed in [2] turned out to be a consequence. The purpose of this article is to propose a local version for the main result in [5].

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**Theorem 1.1.** *Let $D$ be a bounded domain in $\mathbf{C}^2$. Suppose that there exists a point $p \in D$ and a sequence $\{\phi_j\} \in \mathrm{Aut}(D)$ such that $\phi_j(p)$ converges to $p_\infty \in \partial D$. Assume that the boundary of $D$ is smooth real analytic and of finite type near $p_\infty$. Then exactly one of the following alternatives holds:*

(i) *If $\dim \mathrm{Aut}(D) = 2$ then either*
- $D \backsimeq \mathcal{D}_1 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_1(\Re z_1) < 0\}$ *where $P_1(\Re z_1)$ is a polynomial that depends on $\Re z_1$, or*
- $D \backsimeq \mathcal{D}_2 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_2(|z_1|^2) < 0\}$ *where $P_2(|z_1|^2)$ is a polynomial that depends on $|z_1|^2$, or*
- $D \backsimeq \mathcal{D}_3 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_{2m}(z_1, \overline{z}_1) < 0\}$ *where $P_{2m}(z_1, \overline{z}_1)$ is a homogeneous polynomial of degree $2m$ without harmonic terms.*

(ii) *If $\dim \mathrm{Aut}(D) = 3$ then $D \backsimeq \mathcal{D}_4 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + (\Re z_1)^{2m} < 0\}$ for some integer $m \geq 2$.*

(iii) *If $\dim \mathrm{Aut}(D) = 4$ then $D \backsimeq \mathcal{D}_5 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0\}$ for some integer $m \geq 2$. Note that $\Omega_3 \backsimeq E_m$.*

(vi) *If $\dim \mathrm{Aut}(D) = 8$ then $D \backsimeq \mathcal{D}_6 = \mathbf{B}^2$ the unit ball in $\mathbf{C}^2$.*

*The dimensions $0, 1, 5, 6, 7$ cannot occur with $D$ as above.*

While $\mathcal{D}_4, \mathcal{D}_5, \mathcal{D}_6$ are evidently pseudoconvex, no such claim is being made about any of the model domains in case $\dim \mathrm{Aut}(D) = 2$. Indeed, pseudoconvexity is not always assured as the following example from [18] shows. Consider the bounded domain

$$\Omega = \Big\{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^4 + 8|z_1 - 1|^2 \Big(\frac{z_2^2}{z_1 - 1} + \frac{\overline{z}_2^2}{\overline{z}_1 - 1} - \frac{3}{2}\frac{|z_2|^2}{|z_1 - 1|}\Big) < 1\Big\}$$

and set $a_j = 1 - 1/j$ where $j \geq 1$. The sequence

$$\phi_j(z_1, z_2) = \Big(\frac{z_1 - a_j}{1 - a_j z_1}, \frac{(1 - a_j^2)^{1/4} z_2}{1 - a_j z_1}\Big) \in \mathrm{Aut}(\Omega)$$

converges uniformly on compact subsets of $\Omega$ to $(-1, 0) \in \partial\Omega$. Note that the boundary $\partial\Omega$ near $(-1, 0)$ is smooth real analytic and of finite type. The mapping $f(z_1, z_2) = \Big((z_1 + 1)/(z_1 - 1), \sqrt{2} z_2/\sqrt{z_1 - 1}\Big)$ biholomorphically transforms $\Omega$ onto its unbounded realisation given by

$$\Omega' = f(\Omega) = \Big\{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_1 + \frac{1}{4}|z_2|^4 + 2\Big(z_2^2 + \overline{z}_2^2 - \frac{3}{2}|z_2|^2\Big)^2 < 0\Big\}.$$

The terms involving $z_2$ are homogeneous of order 4 and therefore $\Omega'$ admits automorphisms of the form $S_s(z_1, z_2) = (\exp(s) z_1, \exp(s/4) z_2)$ for $s \in \mathbf{R}$. Also, $\Omega'$ is evidently invariant under translations in the imaginary $z_1$-direction and this shows that $\dim \mathrm{Aut}(\Omega') \geq 2$. If $\dim \mathrm{Aut}(\Omega') > 2$, then the above theorem shows that $\Omega'$ and hence $\Omega$ must be pseudoconvex. This however does not hold; for instance, the boundary $\partial\Omega'$ is not pseudoconvex near the point $(-3/4, 1) \in \partial\Omega'$ as a calculation of the Levi form of $\partial\Omega'$ shows.

The two principal techniques used in [2], [5] and [6] are scaling and a careful analysis of a holomorphic tangential vector field of parabolic type. Moreover, the hypotheses that $\partial D$ is globally smooth real analytic (in [2] and [5]) and that $p_\infty$ is a smooth weakly pseudoconvex finite type boundary point in [6] are used in an important way. These hypotheses are not assumed in the theorem above and hence the two techniques have to be supplemented with information regarding the type of orbits that are possible when $\dim \mathrm{Aut}(D) = 3, 4$. This leads to a classification of $D$ that depends on the dimension of $\mathrm{Aut}(D)$.

As mentioned above it is known that $0 \leq \dim \mathrm{Aut}(D) \leq n^2 + 2n = 8$ (since $n = 2$) both inclusive. By a result of W. Kaup (see [28]) $\mathrm{Aut}(D)$ acts transitively on $D$ if $\dim \mathrm{Aut}(D) \geq 5$. Since the boundary of $D$ is smooth real analytic and of finite type near $p_\infty$, it can be shown that there are strongly pseudoconvex points arbitrarily close to $p_\infty$. The Wong-Rosay theorem now shows that $D \backsimeq \mathcal{D}_6 = \mathbf{B}^2$. Hence the remaining possibilities are $0 \leq \dim \mathrm{Aut}(D) \leq 4$. An initial scaling of $D$, as in [5], with respect to the sequence $p_j := \phi_j(p)$ shows that $D$ is equivalent to a domain of the form

$$G_p = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P(z_1, \overline{z}_1) < 0\}$$

where $P(z_1, \overline{z}_1)$ is a polynomial without harmonic terms. Note that $G_p$ is invariant under translations $(z_1, z_2) \mapsto (z_1, z_2 + it), t \in \mathbf{R}$ and hence $\dim \mathrm{Aut}(D) \geq 1$. The possibility that it equals one is ruled out by arguments similar to those in [6] and [5]. Thus $\dim \mathrm{Aut}(D) = 2, 3$ or $4$. When $\dim \mathrm{Aut}(D) = 2$ the

arguments used in [36] can be adapted to show that $D$ is equivalent to $\mathcal{D}_1, \mathcal{D}_2$ or $\mathcal{D}_3$. Finally when the dimension is 3 or 4, the classification obtained in [23], [24] is used. In case $\dim \mathrm{Aut}(D) = 4$ techniques of analytic continuation of germs of holomorphic mappings as in [40] are used to show that $D \backsimeq \mathcal{D}_5$. There are many more possibilities for $D$ in case $\dim \mathrm{Aut}(D) = 3$ as [23] shows. Several on that list have Levi flat orbits foliated by copies of the unit disc in the complex plane. However, it is shown that $D$ as in the theorem cannot admit Levi flat orbits. A further reduction is obtained by studying the Lie algebra of $\mathrm{Aut}(D)$ of some of the examples and showing that $D$ is forced to be equivalent to a tube domain. This uses ideas from [36] and [34]. This does not exhaust the list; the remaining possibilities are ruled out by using arguments that were developed to study the boundary regularity of holomorphic mappings between domains in $\mathbf{C}^n$. The conclusion is that $D \backsimeq \mathcal{D}_4$.

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## 2. The dimension of $\mathrm{Aut}(D)$ is at least two

Let $D$ be as in theorem 1.1 and $p \in D$ and $\phi_j \in \mathrm{Aut}(D)$ are such that $\phi_j(p) \to p_\infty \in \partial D$. Let $U$ be an open neighbourhood of $p_\infty$, fixed henceforth, such that the boundary of $D$ is smooth real analytic and of finite type in a neighbourhood of $\overline{U}$ and that $U \cap \partial D$ is defined by $\{\rho(z, \overline{z}) = 0\}$ with $d\rho \neq 0$ on $U \cap \partial D$ for some $\rho \in C^\omega(U)$. For every $a \in U \cap \partial D$, $T^c_a(\partial D)$ the complex tangent space at $a$ is spanned by the non-vanishing vector field $X_a = (-\partial\rho/\partial z_2(a), \partial\rho/\partial z_1(a))$. The function

$$\mathcal{L}_\rho(z) = \mathcal{L}_\rho(z, X_z)$$

where the term on the right is the Levi form associated with the defining function $\rho$ evaluated at $z \in U \cap \partial D$ and $X_z$, is real analytic on $U \cap \partial D$. This function provides a decomposition of $U \cap \partial D$ which will be useful in this context and we recall its salient features from [14].

Let $T$ be the zero locus of $\mathcal{L}_\rho(z)$ in $U \cap \partial D$. Then $T$ admits a semi-analytic stratification as $T = T_0 \cup T_1 \cup T_2$ where $T_j$ is a locally finite union of smooth real analytic submanifolds of $U \cap \partial D$ of dimension $j = 0, 1, 2$ respectively. Denote by $(U \cap \partial D)^\pm_s$ the set of strongly pseudoconvex (resp. strongly pseudoconcave) points on $U \cap \partial D$. Let $(U \cap \partial D)^\pm$ be the relative interior, taken with respect to the subspace topology on $U \cap \partial D$, of the closure of $(U \cap \partial D)^\pm_s$ in $\overline{U} \cap \partial D$. Then $(U \cap \partial D)^\pm$ is the set of weakly pseudoconvex (resp. weakly pseudoconcave) points on $U \cap \partial D$ and the border

$$\mathcal{B} = (U \cap \partial D) \setminus ((U \cap \partial D)^+ \cup (U \cap \partial D)^-) \subset T$$

separates $(U \cap \partial D)^+$ and $(U \cap \partial D)^-$. The stratification of $T$ can be refined in such a way that the two dimensional strata become maximally totally real and the order of vanishing of $\mathcal{L}_\rho(z)$ along them is constant. The same notation $T_j$ will be retained to denote the various strata after the refinement. It was shown in [12] that if the order of vanishing of $\mathcal{L}_\rho(z)$ along a two dimensional stratum, say $S$ is odd then $S \subset \hat{D}$, the envelope of holomorphy of $D$, while if it is even then $S \subset (U \cap \partial D)^+$ or $S \subset (U \cap \partial D)^- \subset \hat{D}$. This discussion holds for a germ of a smooth real analytic, finite type hypersurface in $\mathbf{C}^2$ and is independent of any assumptions on $\mathrm{Aut}(D)$. The question now is to identify where $p_\infty$ lies in this decomposition of $U \cap \partial D$. First observe that $p_\infty \in T$ as otherwise it is either in $(U \cap \partial D)^+_s$ or $(U \cap \partial D)^-_s$. In the former case, the Wong-Rosay theorem shows that $D \backsimeq \mathbf{B}^2$ while the latter possibility does not arise; indeed it was observed by Greene-Krantz (see [21]) that no point in $\hat{D}$ can be a boundary orbit accumulation point. If $p_\infty \in (U \cap \partial D)^+$ then all possible model domains are known by [6] while $p_\infty \notin (U \cap \partial D)^-$ as all weakly pseudoconcave points are contained in $\hat{D}$. It therefore follows that $p_\infty \in \mathcal{B}$ which means that $\mathcal{L}_\rho(z)$ must change sign in arbitrarily small neighbourhoods of $p_\infty$. Second, if $p_\infty$ belongs to a two dimensional stratum of $T$ then the discussion above shows that $p_\infty$ belongs either to $(U \cap \partial D)^+$ or $\hat{D}$. As before the former case is handled by [6] while the latter does not happen by the Greene-Krantz observation. It is thus possible to assume without loss of generality that $p_\infty \in \mathcal{B} \cap (T_0 \cup T_1)$. This will be the standing assumption henceforth. In sections 4 and 5 it will be shown that $\partial D$ is pseudoconvex near $p_\infty$.

**Lemma 2.1.** *In the situation described above, there exists at least one stratum in $T_2$, say $S$ that contains $p_\infty$ in its closure and for which the order of vanishing of $\mathcal{L}_\rho(z)$ along it is odd. In particular $S \subset \hat{D}$.*

*Proof.* If possible let $V \subset U$ be a neighbourhood of $p_\infty$ such that $V \cap \partial D$ contains no two dimensional stratum of $T$. This implies that $V \cap T$ does not separate $V \cap \partial D$. Choose $a, b \in V \cap \partial D$ such that $\mathcal{L}_\rho(a) > 0$ and $\mathcal{L}_\rho(b) < 0$ and join them by a path $\gamma(t)$ parametrised by $[0,1]$ with the end points corresponding to $a, b$ and which lies entirely in $(V \cap \partial D) \setminus T$. The function $t \mapsto \mathcal{L}_\rho(\gamma(t))$ changes sign and hence $\mathcal{L}_\rho(\gamma(t_0)) = 0$ for some $0 < t_0 < 1$ which means that $\gamma(t_0) \in T$. This is a contradiction. Hence $T_2$ is non-empty near $p_\infty$.

Now fix a ball $B_\epsilon = B(p_\infty, \epsilon)$ for some $\epsilon > 0$. Then $B_\epsilon \setminus T$ has finitely many components each of which contains $p_\infty$ in its closure and the sign of $\mathcal{L}_\rho(z)$ does not change within each component. Let $S_1, S_2, \ldots, S_k$ $(k \geq 1)$ be all the two dimensional strata each of which contains $p_\infty$ in its closure. Note that the union of the $S_j$'s is contained in the union of the boundaries of the various components of $B_\epsilon \setminus T$. Choose $a_j \in S_j$ for all $1 \leq j \leq k$ and let $\sigma(t) : [0,1] \to (B_\epsilon \cap \partial D) \setminus (T_0 \cup T_1)$ be a closed path that contains all the $a_j$'s. Let $0 \leq t_j \leq 1$ be such that $\sigma(t_j) = a_j$ for all $1 \leq j \leq k$. The function $t \mapsto \mathcal{L}_\rho(\sigma(t))$ then has zeros at least at all the $t_j$'s. If $S_j \subset (U \cap \partial D)^+$ for all $1 \leq j \leq k$, then $\mathcal{L}_\rho(\sigma(t))$ is non-negative on $[0,1]$ and hence $p_\infty$ is a weakly pseudoconvex point. This is not possible. There is therefore at least one value, say $j_0$ for which $S_{j_0} \subset (U \cap \partial D)^-$. This implies that $\mathcal{L}_\rho(\sigma(t))$ changes sign on $[0,1]$ which in turn shows the existence of a two dimensional stratum from the collection $S_1, S_2, \ldots, S_k$, say $S_{i_o}$ with the property that $\mathcal{L}_\rho(z)$ changes sign near each point on it. Thus the order of vanishing of $\mathcal{L}_\rho(z)$ along $S_{i_0}$ must be odd. The same argument works when some of the $S_j$'s (though not all) are contained in $(U \cap \partial D)^-$ and the remaining in $(U \cap \partial D)^+$. The case when all the $S_j$'s belong to $(U \cap \partial D)^-$ does not arise because then $p_\infty \in (U \cap \partial D)^-$ which cannot hold by the Greene-Krantz observation above. $\square$

**Lemma 2.2.** *The sequence $\{\phi_j\} \in \mathrm{Aut}(D)$ converges uniformly on compact subsets of $D$ to the constant map $\phi(z) \equiv p_\infty$ for all $z \in D$.*

*Proof.* The family $\{\phi_j\}$ is normal and hence admits a subsequence that converges in the compact open topology on $D$ to $\phi : D \to \overline{D}$ with $\phi(p) = p_\infty$. It then follows from a theorem of H. Cartan (cf. [35]) that $\phi(D) \subset \partial D$. Choose $r > 0$ small enough so that $\phi : B(p, r) \to U$ is well defined. Let $k > 0$ be the maximal rank of $\phi$ which is attained on the complement of an analytic set $A \subset D$. Two cases arise now; first if $p \in D \setminus A$, choose a small ball $B(p, \epsilon)$ which does not intersect $A$. The image $\phi(B(p, \epsilon))$ is then a germ of a complex manifold of dimension $k$ that is contained in $U \cap \partial D$. This cannot happen unless $k = 0$. Second, if $p \in A$ choose $q \in B(p, r) \setminus A$. The rank of $\phi$ is constant near $q$ and hence the image of a small enough neighbourhood of $q$ under $\phi$ is a germ of a complex manifold of dimension $k$ that is contained in $U \cap \partial D$. Again this is not possible. Since $A$ does not separate $B(p, r)$ it follows that $\phi$ is constant on $B(p, r) \setminus A$, therefore on $B(p, r)$ and hence everywhere on $D$. $\square$

*Remark:* It is now possible to conclude (see for example [33]) that $D$ is simply connected. Indeed if $\gamma$ is a loop in $D$ then for $j$ large enough $\phi_j(\gamma)$ is a loop in $U \cap D$ by the above lemma. But $U \cap D$ is simply connected if $U$ is small enough and so $\phi_j(\gamma)$ and hence $\gamma$ (since $\phi_j \in \mathrm{Aut}(D)$) are both trivial loops. This will be useful later.

The domain $D$ can now be scaled using the base point $p$ and the sequence $\{\phi_j\} \in \mathrm{Aut}(D)$. The transformations used in this process are the ones in [5] and are briefly described as follows. First note that for $j$ large there exists a unique point $\tilde{p}_j \in U \cap \partial D$ such that $\mathrm{dist}(\phi_j(p), U \cap \partial D) = |\tilde{p}_j - \phi_j(p)|$. Next translate $p_\infty$ to the origin and rotate axes so that the defining function $\rho(z)$ takes the form

$$\rho(z) = 2\Re z_2 + \sum_{k,l} c_{kl}(y_2) z_1^k \overline{z}_1^l \tag{2.1}$$

where $c_{00}(y_2) = O(y_2^2)$ and $c_{10}(y_2) = \overline{c}_{01}(y_2) = O(y_2)$. Let $m < \infty$ be the 1-type of $\partial D$ at the origin. It follows that there exist $k, l$ both at least one and $k + l = m$ for which $c_{kl}(0) \neq 0$ and $c_{kl}(0) = 0$ for all $k + l < m$. The pure terms in (2.1) up to order $m$ can be removed by a polynomial automorphism of the form

$$(z_1, z_2) \mapsto (z_1, z_2 + \frac{1}{2} \sum_{k \leq m} c_{k0}(0) z_1^k). \tag{2.2}$$

Let $\psi^j_{p,1}(z) = z - \tilde{p}_j$ so that $\psi^j_{p,1}(\tilde{p}_j) = 0$. Next let $\psi^j_{p,2}(z)$ be a unitary transformation that rotates the outer real normal to $\psi^j_{p,1}(U \cap \partial D)$ at the origin and makes it the real $z_2$-axis. The defining function for

$\psi^j_{p,2} \circ \psi^j_{p,1}(U \cap D)$ near the origin is then of the form

$$\rho_j(z) = 2\Re z_2 + \sum_{k,l \geq 0} c^j_{kl}(y_2) z_1^k \overline{z}_1^l$$

with the same normalisations on $c^j_{00}(y_2)$ and $c^j_{10}(y_2)$ as in (2.1). Since $\tilde{p}_j \to 0$ it follows that both $\psi^j_{p,1}$ and $\psi^j_{p,2}$ converge to the identity mapping uniformly on compact subsets of $\mathbf{C}^2$. The 1-type of $\psi^j_{p,2} \circ \psi^j_{p,1}(U \cap \partial D)$ is at most $m$ for all large $j$ and an automorphism of the form (2.2) will remove all the pure terms up to order $m$ from $\rho_j(z)$. Call this $\psi^j_{p,3}$. Finally $\phi_j(p)$ is on the inner real normal to $U \cap \partial D$ at $\tilde{p}_j$ and it follows that $\psi^j_{p,2} \circ \psi^j_{p,1}(\phi_j(p)) = (0, -\delta_j)$ for some $\delta_j > 0$. Furthermore the specific form of (2.2) shows that this is unchanged by $\psi^j_{p,3}$. Let $\psi^j_{p,4}(z_1, z_2) = (z_1/\epsilon_j, z_2/\delta_j)$ where $\epsilon_j > 0$ is chosen in the next step. The defining function for $\psi^j_p(U \cap D)$ near the origin, where $\psi^j_p = \psi^j_{p,4} \circ \psi^j_{p,3} \circ \psi^j_{p,2} \circ \psi^j_{p,1}$, is given by

$$\rho_{j,p}(z) := \frac{1}{\delta_j} \rho_j(\epsilon_j z_1, \delta_j z_2) = 2\Re z_2 + \sum_{k,l} \epsilon_j^{k+l} \delta_j^{-1} c^j_{kl}(\delta_j y_2) z_1^k \overline{z}_1^l.$$

Note that $\psi^j_p \circ \phi_j(p) = (0, -1)$ for all $j$. The choice of $\epsilon_j$ is determined by enforcing

$$\max\{|\epsilon_j^{k+l} \delta_j^{-1} c^j_{kl}(0)|;\ k + l \leq m\} = 1$$

for all $j$. In particular $\{\epsilon_j^m \delta_j^{-1}\}$ is bounded and by passing to a subsequence it follows that $\rho_{j,p}(z)$ converges to $2\Re z_2 + P(z_1, \overline{z}_1)$. Therefore the domains $\psi^j_p(U \cap D)$ converge to

$$G_p = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P(z_1, \overline{z}_1) < 0\} \tag{2.3}$$

in the sense that every compact $K \subset G_p$ is eventually compactly contained in $\psi^j_p(U \cap D)$ and conversely every compact $K \subset \psi^j_p(U \cap D)$ for all large $j$ is compact in $G_p$. The polynomial $P(z_1, \overline{z}_1)$ is of degree at most $m$ and does not have any harmonic terms. The family $g_{j,p} = (\psi^j_p \circ \phi_j)^{-1} : \psi^j_p(U \cap D) \to D$ is normal and the arguments in [5] can be applied in this local setting as well to show that a subsequence converges to a biholomorphic mapping $g_p : G_p \to D$. $G_p$ is invariant under the one parameter group of translations $T_t(z_1, z_2) = (z_1, z_2 + it)$, $t \in \mathbf{R}$ and hence the dimension of $\mathrm{Aut}(D)$ is at least one. For brevity we shall write $g, G$ in place of $g_p, G_p$ respectively.

The holomorphic vector field corresponding to the action of $T_t$ is $i\ \partial/\partial z_2 = i/2\ \partial/\partial x_2 + 1/2\ \partial/\partial y_2$. Then $\mathcal{X} = g_*(i\ \partial/\partial z_2)$ is a holomorphic vector field on $D$ whose real part $\Re\mathcal{X} = (\mathcal{X} + \overline{\mathcal{X}})/2$ generates the one parameter group $L_t = g \circ T_t \circ g^{-1} = \exp(t\ \Re\mathcal{X}) \in \mathrm{Aut}(D)$. In general the Lie algebra $\mathfrak{g}(M)$ of $\mathrm{Aut}(M)$, where $M$ is a Kobayashi hyperbolic complex manifold of dimension $n$, consists of real vector fields, i.e., those of the form

$$\sum_{j \leq n} \Big( a_j(z) \partial/\partial z_j + \overline{a_j(z)} \partial/\partial \overline{z}_j \Big)$$

where the $z_j$ are local coordinates and the coefficients $a_j(z)$ are holomorphic. Such fields are clearly determined by their $(1,0)$ components and therefore it suffices to indicate only these components. This convention shall be followed everywhere in the sequel.

**Proposition 2.3.** *The group $(L_t)$ induces a local one parameter group of holomorphic automorphisms of a neighbourhood of $p_\infty$ in $\mathbf{C}^2$. In particular $\mathcal{X}$ extends as a holomorphic vector field near $p_\infty$.*

It is possible to choose sufficiently small neighbourhoods $p_\infty \in U_1 \subset U_2$ with $U_2$ relatively compact in $U$ such that for each $w \in U_1$, the associated Segre variety $Q_w = \{z \in U_2 : \rho(z, \overline{w}) = 0\}$ is a closed complex hypersurface in $U_2$. Let $\mathcal{S}(U_1, U_2)$ be the aggregate of all Segre varieties and $\lambda : U_1 \to \mathcal{S}(U_1, U_2)$ given by $\lambda(w) = Q_w$. It is known that (see [16], [13]) that $\mathcal{S}(U_1, U_2)$ admits the structure of a finite dimensional complex analytic set and that $\lambda$ is locally an anti-holomorphic finite-to-one branched covering. First assume that $p_\infty \in T_1$. Therefore by shrinking $U$ it follows that $p_\infty$ lies on an embedded real analytic arc, which will still be denoted by $T_1$ and $(U \cap \partial D) \setminus T_1$ consists either of weakly pseudoconvex, finite type points or those that belong to $\hat{D}$. Furthermore $T_1$ admits a complexification, denoted by $T_1^{\mathbf{C}}$, which is a closed, smooth one dimensional analytic set in $U$ (shrink $U$ further if needed) and since $\lambda$ is a finite map near $p_\infty$, there are only finitely many points in $U$ whose Segre varieties coincide with $T_1^{\mathbf{C}}$. To start

with we will then suppose that $Q_{p_\infty} \neq T_1^{\mathbf{C}}$. The remaining cases when $Q_{p_\infty} = T_1^{\mathbf{C}}$ or when $p_\infty \in T_0$ are similar and will need the knowledge of the conclusions obtainable in case $Q_{p_\infty} \neq T_1^{\mathbf{C}}$. Now choose coordinates centered at $p_\infty$ so that $p_\infty = 0$ and the defining function $\rho(z)$ near the origin takes the form

$$\rho(z) = 2\Re z_2 + o(|z|)$$

for $z \in U$. For $w = (w_1, w_2) \in U$, it reflection in $U \cap \partial D$, denoted by ${}^\kappa w$, is the unique point whose first coordinate is $w_1$ and which lies on $Q_w$, i. e., ${}^\kappa w = (w_1, {}^\kappa w_2)$ and $\rho({}^\kappa w, \overline{w}) = 0$. The map $\kappa(w) = {}^\kappa w$ is a real analytic diffeomorphism near the origin that depends on the choice of a coordinate system. Finally a word about notation: for a neighbourhood $\Omega$ of $p_\infty = 0$, let $\Omega^\pm = \{z \in \Omega : \pm\rho(z) > 0\}$ and ${}_{\kappa w}Q_w$ will denote the germ of $Q_w$ at ${}^\kappa w \in Q_w$.

For $\eta > 0$ small and $|t| < \eta$, and a sufficiently small pair of neighbourhoods $U_1 \subset U_2$ that contain the origin define

$$V_t^+ = \{(w, \tilde{w}) \in U_1^+ \times U_1^+ : L_t(Q_w \cap D) \supset {}_{\kappa \tilde{w}}Q_{\tilde{w}}\} \text{ and } V_t^- = \{(w, \tilde{w}) \in U_1^- \times U_1^- : Q_{L_t(w)} = Q_{\tilde{w}}\}.$$

For each fixed $t \in (-\eta, \eta)$, $V_t^+$ is non-empty as there are boundary points arbitrarily close to the origin that belong to $\hat{D}$ and hence $L_t$ extends holomorphically across such points and the invariance property of Segre varieties (which is the defining condition for $V_t^+$) holds. Moreover $V_t^+$ is locally complex analytic near each of its points being the graph of the correspondence $w \mapsto \lambda^{-1} \circ L_t \circ \lambda(w)$, contains the graph of the extension of $L_t$ near points in $\hat{D}$ and is pure two dimensional after removing all components of lower dimension if any. On the other hand $V_t^-$ is the graph of the correspondence $w \mapsto \lambda^{-1} \circ \lambda \circ L_t(w)$ and hence it contains the graph of $L_t$ over a non-empty open set in $U_1^-$. However, note that the projection from $V_t^\pm$ to the first factor $U_1^\pm$ is not known to be globally proper. It is locally proper near each point in $V_t^\pm$ since $\lambda$ is a finite map. Evidently $V_t^-$ is also two dimensional and by their construction it follows that for each fixed $t \in (-\eta, \eta)$ the locally analytic sets $V_t^+, V_t^-$ can be glued together near all points on $U_1 \cap \partial D$ across which $L_t$ holomorphically extends. Let $V_t$ denote the locally complex analytic set that is obtained from $V_t^\pm$ in this manner.

**Lemma 2.4.** *Suppose that $Q_0 \neq T_1^{\mathbf{C}}$. Then it is possible to choose $\eta > 0$ and $U_1 \subset U_2$ small enough so that $V_t^+ \subset U_1^+ \times U_1^+$ is a closed complex analytic set for all $|t| < \eta$.*

*Proof.* There exists a neighbourhood basis of pairs $U_1 \subset U_2$ at the origin such that $\lambda : U_1 \to \mathcal{S}(U_1, U_2)$ is proper and $\kappa(U_1)$ is compactly contained in $U_2$. For such a pair and $w \in U_1^+$ define

$$R'_t(w) = \{\tilde{w} \in U_1^+ : L_t(Q_w \cap D) \supset {}_{\kappa \tilde{w}}Q_{\tilde{w}}\}$$

and

$$R_t(w) = \{q \in Q_w \cap \overline{U}_2 \cap \overline{D} : L_t(q) = {}^\kappa\tilde{w} \text{ for some } \tilde{w} \in R'_t(w)\}.$$

Note that $L_0$ is the identity and hence $R'_0(w) = \lambda^{-1} \circ \lambda(w)$ while $R_0(w) = \kappa \circ \lambda^{-1} \circ \lambda(w)$. By the choice of $U_1, U_2$ it follows that $R_0(w)$ is at a positive distance from $\partial U_2$ uniformly for all $w \in U_1^+$. Now let $U_{1,j}$ shrink to the origin and $t_j \to 0$ and suppose that there are points $w_j \in U_{1,j}^+$ for which $R_{t_j}(w_j)$ has points that cluster at $\partial U_2$. So let $q_j \in R_{t_j}(w_j)$ be such that $L_{t_j}(q_j) = {}^\kappa\tilde{w}_j$ and $q_j \to q_0 \in \partial U_2$. Then $w_j, {}^\kappa\tilde{w}_j \to 0$ and since $Q_{w_j} \to Q_0$ it follows that $q_0 \in Q_0 \cap \partial U_2 \cap \overline{D}$. Two cases arise; first, if $q_0 \in Q_0 \cap \partial U_2 \cap D$, then since $L_t$ converges to $L_0$ uniformly on compact subsets of $D$, it follows that ${}^\kappa\tilde{w}_j = L_{t_j}(q_j) \to L_0(q_0) = q_0 \neq 0$ and this contradicts the fact that ${}^\kappa\tilde{w}_j \to 0$. The remaining possibility is that $q_0 \in Q_0 \cap \partial U_2 \cap \partial D$. Note that $Q_0 \cap \partial U_2 \cap \partial D$ consists either of weakly pseudoconvex, finite type points or those that belong to $\hat{D}$. In case $q_0 \in \hat{D}$ each $L_t$, $t \in \mathbf{R}$ extends to a uniform neighbourhood of $q_0$, the extensions being equicontinuous there, and the same argument as above shows that this leads to a contradiction. If $q_0$ is a weakly pseudoconvex, finite type point then there are local plurisubharmonic peak functions near $q_0$ and this can be used (see [6] or [43]) to show that for small $|t|$, each $L_t$ extends to a uniform neighbourhood of $q_0$. Again this leads to the same contradiction as above. This reasoning shows that $V_t^+$ is closed for small $|t|$. Indeed for $t$ fixed, the main obstacle that possibly prevents $V_t^+$ from being closed is that its defining property may cease to hold in the limit. This happens exactly when points in $R_t(w)$ get arbitrarily close to $\partial U_2$. But it has been shown that this does not happen for a uniform choice of $U_1^+$ and $|t| < \eta$ for small enough $\eta > 0$. ∎

*Proof of Proposition 2.3.* This will be divided into three parts. First, it is shown that $(L_t)$ induces a local one parameter group of holomorphic automorphisms near $p_\infty$ under the assumption that $Q_{p_\infty} \neq T_1^{\mathbf{C}}$. This is then used to show that lemma 2.4 holds even when $Q_{p_\infty} = T_1^{\mathbf{C}}$ or $p_\infty \in T_0$. Finally this in turn is used to show that $(L_t)$ induces a local one parameter group of holomorphic automorphisms near $p_\infty$ even when $Q_{p_\infty} = T_1^{\mathbf{C}}$ or $p_\infty \in T_0$.

The real analyticity of $T_1$ implies that there exists a non-negative, strongly plurisubharmonic function $\tau(z)$ near $p_\infty = 0$ whose zero lous is exactly $U_1 \cap T_1$. Indeed $T_1$ can be locally straightened and so there are coordinates centered at the origin in which $U_1 \cap T_1$ coincides with the imaginary $z_1$-axis. The function $\tau(z) = (\Re z_1)^2 + |z_2|^2$ is then the desired candidate. The sub-level sets $\Omega_r = \{z \in U_1 : \tau(z) < r\}$ are strongly pseudoconvex tubular neighbourhoods of $U_1 \cap T_1$. Fix $r > 0$ so small that $U_1 \setminus \overline{\Omega}_r$ is connected and such that the set of points $(c_1, c_2) \in U_1^+$ for which the slice $\{z_1 = c_1\} \cap U_1$ does not intersect $\overline{\Omega}_r$ is non-empty and open. On the other hand, note that the slice $\{z_1 = c_1\} \cap \overline{\Omega}_r$, if non-empty, is compactly contained in $\{z_1 = c_1\} \cap U_1$. By lemma 2.4, $V_t^+$ is a closed complex analytic set in $U_1^+ \times U_1^+$ for $|t| < \eta$. Define

$$\tilde{V}_t^+ = \{(w, \tilde{w}) \in U_1^+ \times U_1^+ : L_t({}_{\kappa w}Q_w) \subset Q_{\tilde{w}} \cap D\}.$$

Since $(L_t)^{-1} = L_{-t}$ it follows that $\tilde{V}_t^+ = V_{-t}^+$ for all $t$. Moreover $L_t$ extends holomorphically across points $w$ that are close to those in $\hat{D}$ and satisfies

$$L_t({}_{\kappa w}Q_w) \subset Q_{L_t(w)} \cap D.$$

This shows that both $V_t^+$ and $V_{-t}^+$ contain the graph of $L_t$ over an open set in $U_1^+$ and it follows that they coincide. Thus we may write $V_{|t|}^+$ in place of $V_t^+$ for $|t| < \eta$ and take the defining condition to be that for $\tilde{V}_t^+$.

Let $w_0 \in U_1^+$ have the following property: it is possible to choose a ball $B(w_0, \delta)$ (for some positive $\delta$) compactly contained in $U_1^+$ and such that $V_{|t|}^+ \cap ((B(w_0, \delta) \times U_1^+) \neq \emptyset$ for all $|t| < \eta$. Then for perhaps a smaller $\eta > 0$ it follows that $\partial V_{|t|}^+ \cap (B(w_0, \delta) \times \partial U_1^+) = \emptyset$ for all $|t| < \eta$. Indeed suppose that there exist $(w_j, \tilde{w}_j) \in V_{|t_j|}^+ \cap ((B(w_0, \delta) \times U_1^+)$, $|t_j| \to 0$ such that $\tilde{w}_j$ clusters at $\partial U_1^+$. Then

$$L_{t_j}({}_{\kappa w_j}Q_{w_j}) \subset Q_{\tilde{w}_j} \cap D.$$

for all $j$. Let $w_j \to w_\infty \in \overline{B}(w_0, \delta)$ and $\tilde{w}_j \to \tilde{w}_\infty \in \partial U_1^+$. Since $w_j \in B(w_0, \delta)$ which is compactly contained in $U_1^+$, the germs ${}_{\kappa w_j}Q_{w_j}$ move within a compact subset of $U_1^-$ and hence

$$L_{t_j}({}_{\kappa w_j}Q_{w_j}) \to L_0({}_{\kappa w_\infty}Q_{w_\infty}) = {}_{\kappa w_\infty}Q_{w_\infty} \subset Q_{\tilde{w}_\infty} \cap D.$$

This shows that $\lambda(w_\infty) = \lambda(\tilde{w}_\infty)$ which contradicts the fact that $\lambda : U_1 \to \mathcal{S}(U_1, U_2)$ is proper. The set of points $w_0$ for which this reasoning applies is non-empty and open. An example would be all points in $U_1^+$ that lie close to boundary points in $\hat{D}$. Each $L_t$ would be well defined near such points and will satisfy the invariance property of Segre varieties. This shows that for any compact $K^+ \subset U_1^+$, there exists $\eta > 0$ such that the projection $\pi_{|t|}^+ : V_{|t|}^+ \cap (K^+ \times U_1^+) \to K^+$ is proper for all $|t| < \eta$.

Now fix a compact $K^- \subset U_1^-$. Since $V_0^-$ is the graph of the proper correspondence $w \mapsto \lambda^{-1} \circ \lambda(w)$ and $L_t$ converges uniformly on $K^-$ to $L_0(z) \equiv z$, it follows that the projection $\pi_t^- : V_t^- \cap (K^- \times U_1^-) \to K^-$ is proper for all $|t| < \eta$ for a smaller $\eta > 0$ perhaps. Finally note that $(U_1 \cap \partial D) \setminus \Omega_r$ consists entirely of weakly pseudoconvex, finite type points or those that belong to $\hat{D}$. In both cases, each $L_t$ extends to a uniform neighbourhood (see [6] or [43]) of those points for some $|t| < \eta$. Thus after shrinking $U_1$ and $\eta > 0$ suitably it follows that $V_t \subset (U_1 \setminus \overline{\Omega}_r) \times U_1$ is a closed complex analytic set and the projection

$$\pi_t : V_t \to U_1 \setminus \overline{\Omega}_r$$

is proper. Associated with each $|t| < \eta$ are positive integers $m_t, k_t$ and functions $a_\mu(w,t), b_\nu(w,t)$ for $1 \leq \mu \leq m_t, 1 \leq \nu \leq k_t$ all of which are holomorphic in $U_1 \setminus \overline{\Omega}_r$ such that $V_t \subset \hat{V}_t \subset (U_1 \setminus \Omega_r) \times U_1$ where $\hat{V}_t$ is a pure two dimensional complex analytic set described by $P_1(\tilde{w}_1, w) = P_2(\tilde{w}_2, w) = 0$ (here

$w = (w_1, w_2)$ and $\tilde{w} = (\tilde{w}_1, \tilde{w}_2))$ with

$$P_1(\tilde{w}_1, w) = \tilde{w}_1^{m_t} + a_1(w,t)\tilde{w}_1^{m_t - 1} + \ldots + a_{m_t}(w,t),$$
$$P_2(\tilde{w}_2, w) = \tilde{w}_2^{k_t} + b_1(w,t)\tilde{w}_1^{k_t - 1} + \ldots + b_{k_t}(w,t).$$

At this stage, the functions $a_\mu(w,t), b_\nu(w,t)$ are known to be holomorphic in $w$ for each fixed $t$. Their dependence on $t$, let alone any joint regularity in $(w,t)$ is not known. To remedy this situation, first observe that $V_t$ is the union of the graphs of correspondences that involve $\lambda^{-1}, \lambda$ and $L_t$. Since $\lambda$ and $L_t$ are single valued, the multiplicity of $\pi_t$ is the same as that of $\lambda$ and this shows that $m_t \equiv m$ and $k_t \equiv k$ for all $|t| < \eta$. Second, it follows by lemma A1.3 in [8] that the envelope of holomorphy of $U_1 \setminus \overline{\Omega}_r$ is $U_1$. Hence the functions $a_\mu(w,t), b_\nu(w,t)$ extend to $U_1$ for each fixed $t$ and this implies that $\hat{V}_t$ admits a closed analytic continuation to $U_1 \times U_1$. This extension will still be denoted by $\hat{V}_t$ and since the polynomials $P_1(\tilde{w}_1, w), P_2(\tilde{w}_2, w)$ are monic in their first arguments, the projections $\pi_t$ will continue to be proper over $U_1$. Define

$$\Sigma = \bigcup_{z \in U_1 \cap T_1} Q_z$$

which is a closed, three dimensional real analytic set in $U_2$ and is locally foliated by open pieces of Segre varieties at all of its regular points. The complement $U_1 \setminus \Sigma$ is open and hence non-pluripolar. Fix $w_0 \in U_1 \setminus \Sigma$ and a relatively compact neighbourhood $B(w_0, \delta) \subset U_1 \setminus \Sigma$. Note that for any $w \in B(w_0, \delta), Q_w \cap T_1 = \emptyset$. Two cases arise; first if $Q_{w_0} \cap U_2 \cap \partial D = \emptyset$ then on shrinking $U_2$ if necessary it follows that $Q_w \cap U_2 \cap D$ is uniformly relatively oompact in $D$ for all $w \in B(w_0, \delta)$. The action of $L_t$ on $D$ is real analytic and hence for any fixed $w \in B(w_0, \delta)$, $L_t(Q_w \cap D) = L_t \circ \lambda(w)$ is real analytic in $t$. Second if $Q_{w_0} \cap \partial D \neq \emptyset$ then $Q_w \cap T_1 = \emptyset$ implies that $Q_w \cap \partial D \cap U_2$ consists of weakly pseudoconvex finite type points or those that belong to $\hat{D}$ for all $w$ close to $w_0$. As before it follows from [6], [43] that $L_t(Q_w \cap D)$ depends real analytically on $t$ for each fixed $w \in B(w_0, \delta)$. Let $\sigma$ be the branch locus of $\lambda$. Then $\lambda(\sigma)$ is a complex analytic subset of $\mathcal{S}(U_1, U_2)$ of strictly smaller dimension and hence it is possible to move $w_0$ slightly and to shrink $\delta > 0$ if necessary so that $L_t(Q_w \cap D) \cap \lambda(\sigma) = \emptyset$ for all $(w,t) \in B(w_0, \delta) \times (-\eta, \eta)$. Thus the various branches of $\pi_t^{-1}$ over $B(w_0, \delta)$ are well defined holomorphic functions for each fixed $t$. In particular $a_\mu(w,t), b_\nu(w,t)$ depend real analytically on $t$ for each fixed $w \in B(w_0, \delta)$. It follows from theorem 1 in [41] that $a_\mu(w,t), b_\nu(w,t)$ are jointly real analytic in $(w,t) \in U_1 \times (-\eta, \eta)$. Let $\sigma_t$ be the branch locus of $w \mapsto \lambda^{-1} \circ L_t \circ \lambda$. Then $\sigma_t$ is the zero set of a universal polynomial function of the symmetric functions $a_\mu(w,t), b_\nu(w,t)$. Hence $\sigma_t$ also varies real analytically in $t$. We now work in normal coordinates around $p_\infty = 0$, i.e., coordinates in which the defining function for $U \cap \partial D$ becomes

$$\rho(z) = 2\Re z_2 + \sum_{j \geq 0} c_j(z_1, \overline{z}_1)(\Im z_2)^j$$

where the coefficients $c_j(z_1, \overline{z}_1)$ are real analytic and whose complexifications $c_j(z_1, \overline{w}_1)$ satisfy $c_j(z_1, 0) = 0 = c_j(0, \overline{w}_1)$. It is shown in [14] that $\sigma$ enters $D$ and that $\lambda^{-1} \circ L_t \circ \lambda(\sigma_t) \subset \sigma$ for all $|t| < \eta$. By the invariance property of Segre varieties it follows that $\sigma_t$ also enters $D$ for all $|t| < \eta$. As in theorem 7.4 of [14] consider the discs

$$\Delta'_\epsilon = \{(z_1, z_2) : |z_1| < c, z_2 = -\epsilon\}$$

for $c$ small and $\epsilon > 0$. Then $\Delta_\epsilon = (\lambda^{-1} \circ \lambda)(\Delta'_\epsilon)$ is a family of discs in $U_1$ such that $\overline{\Delta}_0 \cap \sigma_0 = \{0\}$. Hence by continuity

$$\Big( \bigcup_{|t| < \eta} \sigma_t \Big) \cap \overline{\Delta}_\epsilon \Subset \Delta_\epsilon$$

for all $0 < \epsilon < \epsilon_0$ and $\eta > 0$ small. This shows that all possible branching occurs only in the interior of $\Delta_\epsilon$. Thus there is a uniform neighbourhood of

$$\bigcup_{0 \leq \epsilon < \epsilon_0} \partial \Delta_\epsilon$$

to which each $L_t$ extends for all $|t| < \eta$. The disc theorem applied to the family $\Delta_\epsilon, 0 < \epsilon < \epsilon_0$ shows that $L_t$ extends to a uniform neighbourhood of $p_\infty$ for all $|t| < \eta$. This finishes the first part of the proof. Assume now that $p_\infty \in T_0$ or $Q_{p_\infty} = T_1^{\mathbf{C}}$. Note that the proof of lemma 2.4 depends on the fact that there is uniform control on $L_t$ at points of $Q_0 \setminus \{0\}$. This is assured by the above arguments and

thus lemma 2.4 holds even when $p_\infty \in T_0$ or $Q_{p_\infty} = T_1^{\mathbf{C}}$. Once again we may argue as above to show the uniform extendability of $L_t$ across $p_\infty$ for all $|t| < \eta$. This completes the proof of proposition 2.3. Incidentally, this localises the main theorem in [43].

We return to the biholomorphic equivalence $g : G \to D$. Consider the sequence $(a_j, b_j) = g^{-1} \circ \phi_j(p) \in G$, which is the pullback of the orbit $\{\phi_j(p)\} \in D$, and let $2\epsilon_j = 2\Re b_j + P(a_j, \overline{a}_j)$. Note that $\epsilon_j < 0$ for all $j$.

**Proposition 2.5.** *Suppose that $|\epsilon_j| > c > 0$ uniformly for all large $j$. Then $\mathcal{X}$ vanishes to finite order at $p_\infty$.*

*Proof.* First note that $g^{-1} \circ \phi_j(p)$ can cluster only at $\partial G$ since $\phi_j(p) \to p_\infty \in \partial D$. The condition that $2\Re b_j + P(a_j, \overline{a}_j)$ is uniformly strictly positive in modulus ensures that $g^{-1} \circ \phi_j(p)$ clusters only at the point at infinity in $\partial G$. For $\tau \in \mathbf{C}$ note that $(a_j, b_j + |\epsilon_j|\tau) \in G$ whenever $\Re\tau < 1$. With $\mathcal{H} = \{\tau \in \mathbf{C} : \Re\tau < 1\}$, the analytic disc $f_j : \mathcal{H} \to D$ where

$$f_j(\tau) = g(a_j, b_j + |\epsilon_j|\tau)$$

is well defined. Note that $f_j(0) = g(a_j, b_j) = \phi_j(p) \to p_\infty$. The integral curves of $i\,\partial/\partial z_2$ passing through $(a_j, b_j)$ are of the form $\gamma_j(t) = (a_j, b_j + it), t \in \mathbf{R}$ and hence $g \circ \gamma_j(t)$ defines the integral curves of $\mathcal{X}$ through $\phi_j(p)$. For $j$ fixed, as $\tau$ varies on the imaginary axis in $\mathcal{H}$, $(a_j, b_j + |\epsilon_j|\tau)$ sweeps out the integral curves of $i\ \partial/\partial z_2$ through $(a_j, b_j)$. Moreover, since $|\epsilon_j| > c > 0$ the image of the interval $(-M, M)$, for any $M > 0$, under the map $t \mapsto (a_j, b_j + i|\epsilon_j|t)$ contains the line segment from $(a_j, b_j - icM)$ to $(a_j, b_j + icM)$ which equals $\gamma_j((-cM, cM))$ for all $j$.

The family $f_j : \mathcal{H} \to D$ is normal and since $f_j(0) \to p_\infty$, there exists a subsequence of $f_j$ that converges uniformly on compact subsets of $\mathcal{H}$ to a holomorphic mapping $f : \mathcal{H} \to \overline{D}$ with $f(0) = p_\infty$. Identify $\mathcal{H}$ with the unit disc $\Delta(0, 1)$ via a conformal map $\psi : \Delta(0, 1) \to \mathcal{H}$ such that $\psi(0) = 0$. Choose $r \in (0, 1)$ such that $f \circ \psi(\Delta(0, r)) \subset U \cap \partial D$ where $U \cap \partial D$ is a smooth real analytic, finite type hypersurface. Suppose that $f$ is non-constant. Then the finite type assumption on $U \cap \partial D$ implies that $f \circ \psi(\Delta(0, r)) \cap D \neq \emptyset$. The strong disk theorem in [44] forces $p_\infty \in \hat{D}$ which is a contradiction. Thus $f(\tau) \equiv p_\infty$ for all $\tau \in \mathcal{H}$. For an arbitrarily small neighbourhood $V$ of $p_\infty$ and the compact interval $[-iM, iM] \subset \mathcal{H}$, this means that $g \circ \gamma_j((-cM, cM)) \subset f_j([-iM, iM]) \subset V$ for all large $j$ which implies that the integral curve of $\mathcal{X}$ through $\phi_j(p)$ is contained in $V$ for all $t \in (-cM, cM)$. Since the $f_j$'s form a normal family, it follows that $g \circ \gamma_j$ converge uniformly on $[-cM, cM]$ to a path $\gamma_\infty : [-cM, cM] \to V \cap \overline{D}$ with $\gamma_\infty(0) = p_\infty$. Since $\mathcal{X}$ is holomorphic in a neighbourhood of $p_\infty$, it follows that $\gamma_\infty$ is the integral curve of $\mathcal{X}$ through $p_\infty$. As $V$ and $M$ are arbitrary the vector field $\mathcal{X}$ must vanish at $p_\infty$. If $\mathcal{X}$ vanishes to infinite order at $p_\infty$ then $L_t = \exp(t\,\Re\mathcal{X})$ shows that the identity mapping and $L_t$, for all $|t| < \eta$, agree to infinite order at $p_\infty$. By proposition 2.3, each $L_t$ for $|t| < \eta$ extends to a uniform neighbourhood of $p_\infty$ and hence $L_t$ is the identity mapping for all small $t$. This is a contradiction since the action of the one-parameter group $(L_t)$ on $D$ does not have fixed points. ☐

**Proposition 2.6.** *Suppose that* $\dim \mathrm{Aut}(G) = 1$ *and* $g \in \mathrm{Aut}(G)$. *Then* $g(z_1, z_2) = (\alpha z_1 + \beta, \phi(z_1) + az_2 + b)$ *where* $\alpha, a \in \mathbf{C} \setminus \{0\}, \beta, b \in \mathbf{C}$ *and* $\phi(z_1)$ *is entire.*

*Proof.* Since $\dim \mathrm{Aut}(G) = 1$ it follows that the connected component of the identity $\mathrm{Aut}(G)^c$ must be the group generated by the translations $T_t(z_1, z_2) = (z_1, z_2 + it)$ and each such $T_t$ evidently has the form mentioned above. So we may suppose that $g \notin \mathrm{Aut}(G)^c$. In this case, note that since $\mathrm{Aut}(G)^c$ is normal in $\mathrm{Aut}(G)$, it follows that for each $t \in \mathbf{R}$ there exists $t' = f(t) \in \mathbf{R}$ such that $g \circ T_t(z_1, z_2) = T_{f(t)} \circ g(z_1, z_2)$. If $g(z_1, z_2) = (g_1(z_1, z_2), g_2(z_1, z_2))$ this is equivalent to

$$\begin{aligned} g_1(z_1, z_2 + it) &= g_1(z_1, z_2), \\ g_2(z_1, z_2 + it) &= g_2(z_1, z_2) + if(t). \end{aligned}$$

The first equation implies that $\partial g_1/\partial z_2 \equiv 0$ and hence $g_1(z_1, z_2) = g_1(z_1)$. Moreover, if $\pi : \mathbf{C}^2 \to \mathbf{C}_{z_1}$ is the natural projection, then $\pi(G) = \mathbf{C}_{z_1}$ as the defining function for $G$ shows. Therefore $g_1(z_1)$ is entire and furthermore $g_1(z_1) \in \mathrm{Aut}(\mathbf{C})$ since this reasoning applies to $g^{-1}$ as well. Hence $g_1(z_1) = \alpha z_1 + \beta$ for

some $\alpha \in \mathbf{C} \setminus \{0\}$. For $g_2$ fix $t \in \mathbf{R}$ arbitrarily and differentiate with respect to $z_1, z_2$. This gives

$$\partial g_2/\partial z_1(z_1, z_2 + it) = \partial g_2/\partial z_1(z_1, z_2),$$
$$\partial g_2/\partial z_2(z_1, z_2 + it) = \partial g_2/\partial z_2(z_1, z_2)$$

both of which hold for all $t \in \mathbf{R}$. From the first equation above it can be seen that $\partial g_2/\partial z_1(z_1, z_2)$ is independent of $z_2$, i.e., $g_2(z_1, z_2) = \phi(z_1) + \psi(z_2)$ for some holomorphic $\phi(z_1), \psi(z_2)$. Putting this in the second equation it follows that $\psi'(z_2 + it) = \psi'(z_2)$ and therefore $\psi(z_2) = az_2 + b$ for some $a, b \in \mathbf{C}$. Now observe that

$$Dg(z_1, z_2) = \begin{pmatrix} \alpha & 0 \\ \phi'(z_1) & a \end{pmatrix}$$

which must be non-singular for all $(z_1, z_2) \in G$ and so $\alpha a \neq 0$. It can then be noted that $g$ is in fact an automorphism of $\mathbf{C}^2$ that stabilises $G$. The iterates of $g$ can also be computed. For $n \geq 1$, let $g^n = g \circ g \circ \cdots \circ g \in \mathrm{Aut}(G)$ be the $n$-fold composition of $g$ with itself. From the explicit form of $g$ it can be seen that $g^n(z_1, z_2) = (\alpha^n z_1 + \beta_n, \phi_n(z_1) + a^n z_1 + b_n)$ where $\beta_n = \beta(1 - \alpha^n)/(1 - \alpha)$, $b_n = b(1 - a^n)/(1 - a)$, and $\phi_n(z_1)$ is entire for all $n \geq 1$. Note that it is possible to compute $\phi_n(z_1)$ explicitly in terms of $\phi(z_1), \alpha, \beta, a, b$ but this shall not be needed. A similar calculation can be done for $n \leq -1$. $\square$

**Proposition 2.7.** *The function $\phi(z_1)$ is a polynomial and $a \in \mathbf{R}$. Moreover $|\alpha| = 1$ and consequently $|a| = 1$.*

*Proof.* Since $g$ extends to an automorphism of $\mathbf{C}^2$, it must preserve $\partial G$ as well. Hence

$$2\Re(\phi(z_1) + az_2 + b) + P(\alpha z_1 + \beta, \overline{\alpha z_1 + \beta}) = 0$$

whenever $2\Re z_2 + P(z_1, \overline{z}_1) = 0$. For $z_1 \in \mathbf{C}$ and $t \in \mathbf{R}$ note that the point $(z_1, -P(z_1, \overline{z}_1)/2 + it) \in \partial G$. Therefore

$$2\Re\Big(\phi(z_1) + a(-P(z_1, \overline{z}_1)/2 + it) + b\Big) + P(\alpha z_1 + \beta, \overline{\alpha z_1 + \beta}) = 0$$

for all $z_1 \in \mathbf{C}$ and $t \in \mathbf{R}$. Let $a = \mu + i\nu$ so that

$$2\Re(\phi(z_1) + b) + 2(-\mu P(z_1, \overline{z}_1)/2 - t\nu) + P(\alpha z_1 + \beta, \overline{\alpha z_1 + \beta}) = 0. \tag{2.4}$$

By comparing the coefficient of $t$ on both sides it follows that $\nu = 0$, i.e., $a = \mu \in \mathbf{R}$. With this (2.4) becomes

$$2\Re(\phi(z_1) + b) - \mu P(z_1, \overline{z}_1) + P(\alpha z_1 + \beta, \overline{\alpha z_1 + \beta}) = 0 \tag{2.5}$$

for all $z_1 \in \mathbf{C}$. It is now possible to identify $\phi(z_1)$; to do this we simply write $z$ in place of $z_1$ and allow $z, \overline{z}$ to vary independently, i.e., consider

$$F(z, w) = \phi(z) + b + \overline{\phi(\overline{w})} + \overline{b} - \mu P(z, w) + P(\alpha z + \beta, \overline{\alpha} w + \overline{\beta})$$

which is holomorphic in $(z, w) \in \mathbf{C}^2$. Now (2.5) shows that $F = 0$ on $\{w = \overline{z}\}$ which is maximally totally real and hence $F \equiv 0$ everywhere by the uniqueness theorem. Putting $w = 0$ and noting that $P(z, 0) \equiv 0$ (since $P(z, \overline{z})$ does not have pure terms) it follows that

$$\phi(z) = -\overline{\phi(0)} - b - \overline{b} - P(\alpha z + \beta, \overline{\beta}) \tag{2.6}$$

which shows that $\phi(z)$ is a polynomial.

This expression for $\phi(z)$ can be used in (2.5) to get

$$\mu P(z, \overline{z}) = P(\alpha z + \beta, \overline{\alpha z + \beta}) - P(\alpha z + \beta, \overline{\beta}) - \overline{P(\alpha z + \beta, \overline{\beta})} - (\phi(0) + b + \overline{\phi(0) + b}) \tag{2.7}$$

for all $z \in \mathbf{C}$. Let $P_j, P_{j\overline{q}}$ denote derivatives of the form $\partial^j P/\partial z^j$ and $\partial^{j+q} P/\partial z^j \partial \overline{z}^q$ respectively. Note that

$$P(\alpha z + \beta, \overline{\alpha z} + \overline{\beta}) = P(\beta, \overline{\beta}) + \sum_{j \geq 1} \Big(\alpha^j P_j(\beta, \overline{\beta}) \frac{z^j}{j!} + \overline{\alpha}^j \overline{P_j(\beta, \overline{\beta})} \frac{\overline{z}^j}{j!}\Big) + \sum_{j,q>0} P_{j\overline{q}}(\beta, \overline{\beta}) \alpha^j \overline{\alpha}^q \frac{z^j \overline{z}^q}{j! q!}$$

the right side of which will be written as $P(\beta, \overline{\beta}) + I + II$ where $I$ consists of all the pure terms and $II$ contains only mixed terms of the form $z^j \overline{z}^q$ with both $j, q > 0$. Putting this in (2.7) gives

$$\mu P(z, \overline{z}) = P(\beta, \overline{\beta}) + I + II - P(\alpha z + \beta, \overline{\beta}) - \overline{P(\alpha z + \beta, \overline{\beta})} - (\phi(0) + b + \overline{\phi(0) + b}).$$

The left side above has no harmonic terms and therefore

$$P(\beta,\overline{\beta})+I-P(\alpha z+\beta,\overline{\beta})-\overline{P(\alpha z+\beta,\overline{\beta})}-(\phi(0)+b+\overline{\phi(0)+b})\equiv 0$$

which shows that

$$\mu P(z,\overline{z})=\sum_{j,q>0}P_{j\overline{q}}(\beta,\overline{\beta})\alpha^j\overline{\alpha}^q\frac{z^j\overline{z}^q}{j!q!}. \tag{2.8}$$

The point to note now is that in deriving (2.8) only the observation that $g$ stabilises the boundary of $G$ was used and no special properties of $\phi(z)$ played a role. Exactly the same reasoning can therefore be applied to $g^m$, which also stabilises $\partial G$, and this shows that

$$\mu^m P(z,\overline{z})=\sum_{j,q>0}P_{j\overline{q}}(\beta_m,\overline{\beta}_m)(\alpha^m)^j(\overline{\alpha}^m)^q\frac{z^j\overline{z}^q}{j!q!} \tag{2.9}$$

for all $m\in\mathbf{Z}$. Note that if $P(z,\overline{z})$ is homogeneous of degree $l>0$ then $G$ will admit a one-parameter subgroup $s\mapsto S_s(z_1,z_2)=(\exp(s/l)z_1,\exp(s)z_2), s\in\mathbf{R}$ in addition to the translations $T_t$ and hence $\dim\operatorname{Aut}(G)\geq 2$. Therefore, let

$$P(z,\overline{z})=P^{J_1}(z,\overline{z})+P^{J_2}(z,\overline{z})+\ldots+P^{J_n}(z,\overline{z})$$

be the decomposition of $P(z,\overline{z})$ into homogeneous summands of degree $J_i>0$ $(1\leq i\leq n)$ where $J_1<J_2<\ldots<J_n$. Let

$$P^{J_i}(z,\overline{z})=\sum_{k+l=J_i}C^{J_i}_{k\overline{l}}\; z^k\overline{z}^l$$

for $1\leq i\leq n$. Two cases need to be considered:

*Case 1:* When $\beta=0$ rewrite (2.5) as

$$\mu P(z,\overline{z})=2\Re(\phi(z)+b)+P(\alpha z,\overline{\alpha z})$$

and observe that the left side does not have harmonic terms. Therefore $\Re(\phi(z)+b)\equiv 0$. Pick pairs of positive indices $(k,l),(j,q)$ such that $k+l=J_n, j+q=J_{n-1}$ and for which $C^{J_{n-1}}_{j\overline{q}}, C^{J_n}_{k\overline{l}}$ are both non-zero. Comparing the coefficients of $z^k\overline{z}^l$ and $z^j\overline{z}^q$ in (2.8) gives

$$\mu=\alpha^k\overline{\alpha}^l=\alpha^j\overline{\alpha}^q$$

which implies that

$$|\mu|=|\alpha|^{k+l}=|\alpha|^{J_n}=|\alpha|^{j+q}=|\alpha|^{J_{n-1}}.$$

Since $J_{n-1}<J_n$ it follows that $|\alpha|=1$ and consequently $|\mu|=1$.

*Case 2:* Suppose that $\beta\neq 0$. Pick $j,q>0$ such that $j+q=J_{n-1}$ and for which $C^{J_{n-1}}_{j\overline{q}}\neq 0$. Compare the coefficient of $z^j\overline{z}^q$ in (2.9) to get

$$\begin{aligned}\mu^m C^{J_{n-1}}_{j\overline{q}}&=P_{j\overline{q}}(\beta_m,\overline{\beta}_m)\frac{(\alpha^m)^j(\overline{\alpha}^m)^q}{j!q!},\\ &=(\alpha^m)^j(\overline{\alpha}^m)^q C^{J_{n-1}}_{j\overline{q}}+P^{J_n}_{j\overline{q}}(\beta_m,\overline{\beta}_m)\frac{(\alpha^m)^j(\overline{\alpha}^m)^q}{j!q!}\end{aligned} \tag{2.10}$$

for all $m\in\mathbf{Z}$. To obtain a contradiction, assume that $0<|\alpha|<1$ and consider the above equation for $m\geq 1$. Similar arguments will work in the case $|\alpha|>1$ by considering $m\leq -1$. The argument again splits into two parts.

*Sub-Case (a):* Suppose that $P^{J_n}(z,\overline{z})=C^{J_n}_{\sigma\overline{\sigma}}z^\sigma\overline{z}^\sigma$. Comparing the coefficient of $z^\sigma\overline{z}^\sigma$ in (2.8) shows that $\mu=\alpha^\sigma\overline{\alpha}^\sigma=|\alpha|^{2\sigma}$. Using this and recalling that $\beta_m=\beta(1-\alpha^m)/(1-\alpha)$ allows (2.10) to be rewritten as

$$(E_m):\qquad \Big(1-(\alpha^m)^{\sigma-j}(\overline{\alpha}^m)^{\sigma-q}\Big)C^{J_{n-1}}_{j\overline{q}}+C^{J_n}_{\sigma\overline{\sigma}}(*)\Big(\frac{\beta}{1-\alpha}\Big)^{J_n-J_{n-1}}(1-\alpha^m)^{\sigma-j}(1-\overline{\alpha}^m)^{\sigma-q}=0$$

for all $m \geq 1$, where $(*)$ denotes an unimportant but non-zero constant depending on $\sigma, j, q$. Then $(E_m)$ forms an infinite system of linear equations in the unknowns $C^{J_{n-1}}_{j\overline{q}}$ and $C^{J_n}_{j\overline{q}}$. The rank of any pair $(E_m), (E_{m'})$ cannot be two as otherwise both $C^{J_{n-1}}_{j\overline{q}}$ and $C^{J_n}_{\sigma\overline{\sigma}}$ will vanish which cannot be true. Therefore

$$\begin{vmatrix} 1-(\alpha^m)^{\sigma-j}(\overline{\alpha}^m)^{\sigma-q} & (1-\alpha^m)^{\sigma-j}(1-\overline{\alpha}^m)^{\sigma-q} \\ 1-(\alpha^{2m})^{\sigma-j}(\overline{\alpha}^{2m})^{\sigma-q} & (1-\alpha^{2m})^{\sigma-j}(1-\overline{\alpha}^{2m})^{\sigma-q} \end{vmatrix} = 0$$

for all $m \geq 1$. After removing the non-vanishing factors $1-(\alpha^m)^{\sigma-j}(\overline{\alpha}^m)^{\sigma-q}$ and $(1-\alpha^m)^{\sigma-j}(1-\overline{\alpha}^m)^{\sigma-q}$ from the columns respectively, it follows that

$$(1+\alpha^m)^{\sigma-j}(1+\overline{\alpha}^m)^{\sigma-q} - 1 - (\alpha^m)^{\sigma-j}(\overline{\alpha}^m)^{\sigma-q} = 0 \tag{2.11}$$

for all $m \geq 1$. Hence the complex valued function

$$f(z,\overline{z}) = (1+z)^{\sigma-j}(1+\overline{z})^{\sigma-q} - (1+z^{\sigma-j}\overline{z}^{\sigma-q})$$

vanishes at $\alpha^m$ for all $m \geq 1$. However it can be seen that

$$\begin{aligned} f(z,\overline{z}) &= (\sigma-j)z + (\sigma-q)\overline{z} + \ldots \\ &= \Big(x(J_n - J_{n-1}) + \ldots\Big) + i\Big(y(q-j) + \ldots\Big) \end{aligned}$$

where the lower dots indicate terms of higher order and $z = x+iy$. In case $j \neq q$ it follows that the real and imaginary parts of $f(z,\overline{z})$ vanish along smooth real analytic arcs that intersect at the origin transversally. Hence the origin is an isolated zero of $f(z,\overline{z})$. On the other hand $|\alpha|^m \to 0$ as $m \to \infty$ and this forces $\alpha = 0$ which is a contradiction. In case $j = q$, (2.11) becomes

$$|1+\alpha^m|^{2(\sigma-j)} = 1 + |\alpha^m|^{2(\sigma-j)} \tag{2.12}$$

for all $m \geq 1$. This leads to the consideration of the real analytic germ at the origin defined by

$$A = \{z \in \mathbf{C} : |1+z|^{2(\sigma-j)} - 1 - |z|^{2(\sigma-j)} = 0\}.$$

By looking at the lowest order terms, it follows that $A$ is smooth at the origin. Fix a small disc $\Delta(0,\epsilon)$ so that $A \cap \Delta(0,\epsilon)$ is a smooth real analytic arc and define $\theta : A \cap \Delta(0,\epsilon) \to S^1$ by $\theta(z) = z/|z|$. When $\sigma - j = 1$, it can be seen that $A$ is just the $y$-axis and therefore the range of $\theta$ is a two point set on $S^1$. When $\sigma - j > 1$ the smoothness of $A$ implies that the range of $\theta$ is the disjoint union of two open arcs of total length strictly less than $2\pi$. Since $\alpha^m \in A \cap \Delta(0,\epsilon)$ for $m \gg 1$ it follows that $\alpha/|\alpha|$ must be a root of unity as otherwise the set $\{\alpha^m/|\alpha|^m : m \geq 1\}$, which is contained in the range of $\theta$, will be dense in $S^1$. So let $\alpha^\eta \in \mathbf{R}$ for some positive integer $\eta$. Now let $m$ range over all integral multiples of $\eta$ in (2.12) and this gives

$$\Big(1 + (\alpha^\eta)^m\Big)^{2(\sigma-j)} = 1 + \Big((\alpha^\eta)^m\Big)^{2(\sigma-j)}$$

for all $m \geq 1$. The polynomial $(1+\delta)^{2(\sigma-j)} = 1 + \delta^{2(\sigma-j)}$ in the real indeterminate $\delta$ hence has infinitely many roots given by $(\alpha^\eta)^m$ for all $m \geq 1$. It follows that $(\alpha^\eta)^{m_1} = (\alpha^\eta)^{m_2}$ for some $m_1 \neq m_2$ and this shows that either $|\alpha| = 0$ or 1 which is a contradiction.

*Sub-Case (b):* When $P^{J_n}(z,\overline{z})$ is no longer a monomial such as the one considered above, pick $k, l > 0, k \neq l$ such that $k + l = J_n$ and for which $C^{J_n}_{k\overline{l}} \neq 0$. Compare the coefficients of $z^k\overline{z}^l$ and $z^l\overline{z}^k$ in (2.8) to get $\mu = \alpha^k\overline{\alpha}^l = \alpha^l\overline{\alpha}^k$. If $\alpha = |\alpha|e^{i\theta}$ it follows that $\theta \in 2\pi\mathbf{Q}$, i.e., $\alpha/|\alpha|$ is a root of unity. Let $r = |\alpha|$. Now choose $j, q > 0$ with $j + q = J_{n-1}$ such that $C^{J_{n-1}}_{j\overline{q}} \neq 0$ and assume that $|\alpha| \neq 1$. Consider (2.10) for $m \geq 1$. Since $\mu = \alpha^k\overline{\alpha}^l$ and $\beta_m = \beta(1-\alpha^m)/(1-\alpha)$, it can be rewritten as

$$\Big(1 - \Big(\frac{\mu}{\alpha^j\overline{\alpha}^q}\Big)^m\Big)C^{J_{n-1}}_{j\overline{q}} + \sum_{\sigma,\tau} C^{J_n}_{\sigma\overline{\tau}}(*)\Big(\frac{\beta}{1-\alpha}\Big)^{\sigma-j}\Big(\frac{\overline{\beta}}{1-\overline{\alpha}}\Big)^{\tau-q}(1-\alpha^m)^{\sigma-j}(1-\overline{\alpha})^{\tau-q} = 0$$

for all $m \geq 1$, where the $(*)$'s are non-zero constants that depend on $\sigma, \tau, j, q$ and the sum extends over only those pairs $(\sigma,\tau)$ for which $\sigma \geq j$ and $\tau \geq q$. If $\theta = 2\pi(\delta/\eta)$ for relatively prime integers $\delta, \eta$, we

may let $m$ vary over all the positive multiples of $\eta$ to get

$$(E'_m): \qquad \Big(1-(r^\eta)^{m(J_n-J_{n-1})}\Big)C^{J_{n-1}}_{j\overline{q}} + \Big(1-(r^\eta)^m\Big)^{J_n-J_{n-1}} \underbrace{\Big(\sum_{\sigma,\tau} C^{J_n}_{\sigma\overline{\tau}}(*)\Big(\frac{\beta}{1-\alpha}\Big)^{\sigma-j}\Big(\frac{\overline{\beta}}{1-\overline{\alpha}}\Big)^{\tau-q}\Big)}_{C} = 0$$

for all $m \geq 1$. As above, the $(E'_m)$ form an infinite linear system in the unknowns $C^{J_{n-1}}_{j\overline{q}}$ and $C$. The rank of the pair $(E'_m)$ and $(E'_{2m})$ must be less than two for all $m \geq 1$ as otherwise $C^{J_{n-1}}_{j\overline{q}} = 0$ in particular, which is false by assumption. Therefore

$$\begin{vmatrix} 1-(r^\eta)^{m(J_n-J_{n-1})} & (1-(r^\eta)^m)^{J_n-J_{n-1}} \\ 1-(r^\eta)^{2m(J_n-J_{n-1})} & (1-(r^\eta)^{2m})^{J_n-J_{n-1}} \end{vmatrix} = 0$$

for all $m \geq 1$. Factor the non-vanishing terms $1-(r^\eta)^{m(J_n-J_{n-1})}$ and $(1-(r^\eta)^m)^{J_n-J_{n-1}}$ from the columns and this implies

$$1 + r^{\eta m(J_n-J_{n-1})} - \Big(1+(r^{\eta m})\Big)^{J_n-J_{n-1}} = 0$$

for all $m \geq 1$. The reasoning used in sub-case (a) now shows that $r = 0$ or $1$ which is a contradiction.

To conclude, pick $k, l > 0$ such that $k + l = J_n$ (where possibly $k = l$) and compare the coefficient of $z^k\overline{z}^l$ in (2.8) to get $\mu = \alpha^k\overline{\alpha}^l$ and thus $|a| = |\mu| = |\alpha|^{k+l} = 1$.

□

**Lemma 2.8.** *Suppose that* $\dim \mathrm{Aut}(G) = 1$. *Let* $g(z_1, z_2) = (g_1(z_1, z_2), g_2(z_1, z_2)) = (\alpha z_1 + \beta, \phi(z_1) + az_2 + b) \in \mathrm{Aut}(G)$ *and* $q = (q_1, q_2) \in G$ *be an arbitrary point. Define*

$$E = 2\Re(g_2(q_1, q_2)) + P(g_1(q_1, q_2), \overline{g_1(q_1, q_2)}).$$

*Then $E$ is independent of the parameters involved in $g(z_1, z_2)$ and in fact* $|E| = |2\Re q_2 + P(q_1, \overline{q}_1)|$.

*Proof.* Observe that

$$\begin{aligned} E &= 2\Re(\phi(q_1) + aq_2 + b) + P(\alpha q_1 + \beta, \overline{\alpha q_1 + \beta}) \\ &= 2a\Re q_2 + 2\Re(\phi(q_1) + b) + P(\alpha q_1 + \beta, \overline{\alpha q_1 + \beta}) \\ &= 2a\Re q_2 + aP(q_1, \overline{q}_1) \end{aligned}$$

where the second equality uses the fact that $a \in \mathbf{R}$ while the third follows from (2.5). It remains to note that $|a| = 1$ and this completes the proof. □

**Proposition 2.9.** *The dimension of* $\mathrm{Aut}(D)$ *is at least two.*

*Proof.* Suppose not. Then $\dim \mathrm{Aut}(D) = \dim \mathrm{Aut}(G) = 1$. Observe that $(a_j, b_j) = g^{-1} \circ \phi_j(p) = g^{-1} \circ \phi_j \circ g(g^{-1}(p))$ where $g^{-1} \circ \phi_j \circ g \in \mathrm{Aut}(G)$ for all $j \geq 1$. Let $g^{-1}(p) = q = (q_1, q_2) \in G$. It follows from propositions 2.6, 2.7 and lemma 2.8 that

$$|2\Re b_j + P(a_j, \overline{a}_j)| = |2\Re q_2 + P(q_1, \overline{q}_1)| > 0$$

for all $j$. This also shows that $\{g^{-1} \circ \phi_j(p)\}$ can cluster only at the point at infinity in $\partial G$. Now proposition 2.5 shows that $\mathcal{X}(p_\infty) = 0$ and by lemma 3.5 in [5] the intersection of the zero set of $\mathcal{X}$ with $\partial D$ contains $p_\infty$ as an isolated point. Moreover $\mathcal{X}$ does not vanish in $D$ as the action of $(L_t)$ on $D$ does not have fixed points. The sequence $g^{-1} \circ \phi_j(p)$ converges to the point at infinity in $\partial G$ and its image under $g$, namely $\phi_j(p)$ converges to $p_\infty$. Since the cluster set of the point at infinity in $\partial G$ under $g$ is connected it must equal $\{p_\infty\}$. By defining $g(\infty) = p_\infty$, the mapping $g : G \cup \{\infty\} \to D \cup \{p_\infty\}$ is therefore continuous. Now

$$L_t(p) = L_t \circ g(q_1, q_2) = g \circ T_t(q_1, q_2) = g(q_1, q_2 + it)$$

shows that $\lim_{|t|\to\infty} L_t(p) = p_\infty$. For any other $p' \in D$ and $g^{-1}(p') = (q'_1, q'_2) \in G$ it follows that

$$L_t(p') = L_t \circ g(q'_1, q'_2) = g \circ T_t(q'_1, q'_2) = g(q'_1, q'_2 + it)$$

where $g(q'_1, q'_2 + it) \to p_\infty$ since $|(q'_1, q'_2 + it)| \to \infty$. Thus $\lim_{|t|\to\infty} L_t(p') = p_\infty$. This establishes the parabolicity of the action of $(L_t)$ on $D$ and now the arguments in [5] show that $D \backsimeq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$ for some integer $m \geq 1$. This shows that $\dim \mathrm{Aut}(D) = 4$ which is a contradiction. □

## 3. Model domains when $\mathrm{Aut}(D)$ is two dimensional

Since $g : G \to D$ is biholomorphic and $G$ is invariant under the translations $T_t(z_1, z_2) = (z_1, z_2+it), t \in \mathbf{R}$, it follows that if $\mathrm{Aut}(D)^c$ is abelian then it must be isomorphic to $\mathbf{R}^2$ or $\mathbf{R} \times S^1$. The model domains corresponding to these cases can be computed as follows.

**Proposition 3.1.** *Let $D$ be as in the main theorem and suppose that $\dim \mathrm{Aut}(D) = 2$. If $\mathrm{Aut}(D)^c$ is abelian then either $D \backsimeq \mathcal{D}_1 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_1(\Re z_1) < 0\}$ or $D \backsimeq \mathcal{D}_2 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_2(|z_1|^2) < 0\}$ for some polynomials $P_1(\Re z_1)$ and $P_2(|z_1|^2)$ that depend only on $\Re z_1$ and $|z_1|^2$ respectively.*

*Proof.* The hypotheses imply that $\mathrm{Aut}(G)^c$ is two dimensional. Choose a one-parameter subgroup of the form $S_s(z_1, z_2) = (S^1_s, S^2_s)$ that along with $T_t(z_1, z_2) = (z_1, z_2 + it), t \in \mathbf{R}$ generates $\mathrm{Aut}(G)^c$. Since $\mathrm{Aut}(G)^c$ is abelian, it follows that $S_s \circ T_t = T_t \circ S_s$ which in turn implies that

$$S^1_s(z_1, z_2 + it) = S^1_s(z_1, z_2), \text{ and}$$
$$S^2_s(z_1, z_2 + it) = S^2_s(z_1, z_2) + it$$

for all $s, t \in \mathbf{R}$. The first equation above shows that $\partial S^1_s/\partial z_2 \equiv 0$ in $G$ which means that $S^1_s(z_1, z_2)$ is independent of $z_2$, i.e., $S^1_s(z_1, z_2) = S^1_s(z_1)$ for all $s$. Now note that the projection $\pi(z_1, z_2) = z_1$ maps $G$ surjectively onto the $z_1$-axis and this implies that $(S^1_s(z_1)) \subset \mathrm{Aut}(\mathbf{C})$ is a non-trivial one-parameter subgroup. The second equality above shows that $\partial S^2_s/\partial z_2 \equiv 1$ in $\Omega$, i.e., $S^2_s(z_1, z_2) = z_2 + h(s, z_1)$ where $h(s, z_1)$ is an entire function in $z_1$ for all $s \in \mathbf{R}$. Moreover note that $h(0, z_1) = 0$. Thus $S_s$ is an automorphism of $\mathbf{C}^2$ for all $s$. But since $S_s \in \mathrm{Aut}(G)$ it follows that

$$\Re(S^2_s(z_1, z_2)) + P(S^1_s(z_1), \overline{S}^1_s(z_1)) = 0$$

whenever $\Re z_2 = -P(z_1, \overline{z}_1)$. Hence

$$\Re h(s, z_1) = P(z_1, \overline{z}_1) - P(S^1_s(z_1), \overline{S}^1_s(z_1))$$

for all $z_1 \in \mathbf{C}$. It follows that $h(s, z_1)$ is a polynomial for all $s \in \mathbf{R}$.

Returning to the one-parameter subgroup $(S^1_s(z_1)) \subset \mathrm{Aut}(\mathbf{C})$, it is possible to make an affine change of coordinates in $z_1$ so that for all $s \in \mathbf{R}$, $S^1_s(z_1) = z_1 + is$ or $S^1_s(z_1) = \exp(\alpha s) z_1$ for some $\alpha \in \mathbf{C} \setminus \{0\}$. This can be done using the known description of $\mathrm{Aut}(\mathbf{C})$. Two cases arise, the first being

$$S_s(z_1, z_2) = (S^1_s(z_1), S^2_s(z_1, z_2)) = (z_1 + is, z_2 + h(s, z_1))$$

for all $s \in \mathbf{R}$ which implies that

$$h(s_1 + s_2, z_1) = h(s_2, z_1) + h(s_1, z_1 + is_2)$$

for all $s_1, s_2 \in \mathbf{R}$. Writing this as

$$(h(s_1 + s_2, z_1) - h(s_2, z_1))/s_1 = h(s_1, z_1 + is_2)/s_1$$

for non-zero $s_1$ we get that $\partial h/\partial s(s_2, z_1) = \partial h/\partial s(0, z_1 + is_2)$ for all $s_2$. Since $h(s, z_1)$ is a polynomial for all $s$ it follows that $\partial h/\partial s(0, z_1 + is)$ is a polynomial as well. Integration gives

$$h(s, z_1) = q(z_1 + is) + C$$

for some polynomial $q(z)$ and since $h(0, z_1) = 0$, it follows that $h(s, z_1) = q(z_1 + is) - q(z_1)$. Hence

$$S_s(z_1, z_2) = (z_1 + is, z_2 + q(z_1 + is) - q(z_1))$$

for all $s \in \mathbf{R}$. The automorphism $(z_1, z_2) \mapsto (\tilde{z}_1, \tilde{z}_2) = (z_1, z_2 - q(z_1))$ maps $G$ biholomorphically to $\tilde{G} = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re \tilde{z}_2 + Q(\tilde{z}_1, \overline{\tilde{z}_1}) < 0\}$ for some polynomial $Q(\tilde{z}_1, \overline{\tilde{z}_1})$ and conjugates the action of $S_s$ to the automorphism of $\tilde{G}$ given by

$$(\tilde{z}_1, \tilde{z}_2) \mapsto (\tilde{z}_1 + is, \tilde{z}_2).$$

This shows that $Q(\tilde{z}_1, \overline{\tilde{z}_1})$ is invariant under the map $\tilde{z}_1 \mapsto \tilde{z}_1 + is$ and hence $Q(\tilde{z}_1, \overline{\tilde{z}_1}) = Q(\Re z_1)$. This realizes $G$ as a tube domain after a global change of coordinates.

The other case to consider is when

$$S_s(z_1, z_2) = (S^1_s(z_1), S^2_s(z_1, z_2)) = (\exp(\alpha s) z_1, z_2 + h(s, z_1))$$

for some $\alpha \in \mathbf{C} \setminus \{0\}$. As above $h(s, z_1)$ is a polynomial for all $s \in \mathbf{R}$ and hence $S_s(z_1, z_2) \in \mathrm{Aut}(\mathbf{C}^2)$ that preserves $G$. This means that

$$\Re(z_2 + h(s, z_1)) + P(\exp(\alpha s)z_1, \exp(\overline{\alpha} s)\overline{z}_1) = \Re z_2 + P(z_1, \overline{z}_1)$$

for all $s \in \mathbf{R}$ which shows that

$$\Re(h(s, z_1)) + P(\exp(\alpha s)z_1, \exp(\overline{\alpha} s)\overline{z}_1) = P(z_1, \overline{z}_1).$$

Since the right side has no harmonic terms it follows that $\Re(h(s, z_1)) \equiv 0$ for all $s \in \mathbf{R}$ and hence $h(s, z_1) \equiv i\beta s$ for some real $\beta$. Therefore $P(\exp(\alpha s)z_1, \exp(\overline{\alpha} s)\overline{z}_1) = P(z_1, \overline{z}_1)$ for all $s \in \mathbf{R}$. This forces $\alpha = i\gamma$ for some non-zero real $\gamma$ and that $P(z_1, \overline{z}_1)$ must consist only of terms of the form $|z_1|^{2k}$ for some integer $k \geq 1$. Thus $P(z_1, \overline{z}_1) = P(|z_1|^2)$.

Thus if $\mathrm{Aut}(D)^c$ is two dimensional and abelian the following dichotomy holds:

- $D \backsimeq \mathcal{D}_1 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_1(\Re z_1) < 0\}$ and $\mathrm{Aut}(\mathcal{D}_1)^c$ is generated by $T_t(z_1, z_2) = (z_1, z_2 + it)$ and $S_s(z_1, z_2) = (z_1 + is, z_2)$ for $s, t \in \mathbf{R}$.
- $D \backsimeq \mathcal{D}_2 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_2(|z_1|^2) < 0\}$ and $\mathrm{Aut}(\mathcal{D}_2)^c$ is generated by $T_t(z_1, z_2) = (z_1, z_2 + it)$ and $S_s(z_1, z_2) = (\exp(i\gamma s)z_1, z_2 + i\beta s)$ for $s, t \in \mathbf{R}$ and $\beta, \gamma \neq 0$.

□

**Corollary 3.2.** *Suppose that* $\dim \mathrm{Aut}(D) = 2$*. If* $\phi_j \in \mathrm{Aut}(D)^c$ *for all large* $j$*, then* $\mathrm{Aut}(D)^c$ *cannot be abelian.*

*Proof.* If $\mathrm{Aut}(D)^c$ is abelian then the model domains and the generators for the connected component of the identity of the automorphism group are given above. Let $g_1 : D \to \mathcal{D}_i$, $i = 1, 2$ be the biholomorphic equivalences. The sequence $\{g_i \circ \phi_j(p)\}_{j \geq 1}$ then clusters only at $\partial\mathcal{D}_i$ and in fact only at the point at infinity in $\partial\mathcal{D}_i$. To show this let us consider $g_1 : D \to \mathcal{D}_1$. For each $j \gg 1$, $g_1 \circ \phi_j \circ g_1^{-1} \in \mathrm{Aut}(\mathcal{D}_1)^c$ and thus there are unique reals $s_j, t_j$ such that $g_1 \circ \phi_j \circ g_1^{-1} = S_{s_j} \circ T_{t_j}$. If $g_1(p) = (a, b) \in \mathcal{D}_1$ then

$$g_1 \circ \phi_j(p) = S_{s_j} \circ T_{t_j}(a, b) = (a + is_j, b + it_j)$$

for all $j \gg 1$ and since $\{\phi_j\}$ is non-compact at least one of $|s_j|$ or $|t_j| \to \infty$. Hence $|g_1 \circ \phi_j(p)| \to \infty$. Moreover note that

$$|2\Re(b + it_j) + P_1(\Re(a + is_j))| = |2\Re b + P_1(\Re a)| > 0$$

for all $j$. Similarly for $g_2 : D \to \mathcal{D}_2$ we get

$$g_2 \circ \phi_j(p) = S_{s_j} \circ T_{t_j}(a, b) = (\exp(i\gamma s_j)a, b + it_j + i\beta s_j)$$

for all $j \gg 1$. In this case $|t_j| \to \infty$ and thus $|g_2 \circ \phi_j(p)| \to \infty$. Moreover

$$|2\Re(b + it_j + i\beta s_j) + P_2(|\exp(i\gamma s_j)a|^2)| = |2\Re b + P_2(|a|^2)| > 0$$

for all $j$. The arguments used in proposition 2.5 and 2.9 now show that $D \backsimeq \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$ for some integer $m \geq 1$ which means that $\dim \mathrm{Aut}(D) = 4$. This is a contradiction. □

**Proposition 3.3.** *Let* $\dim \mathrm{Aut}(D) = 2$ *and suppose that* $\mathrm{Aut}(D)^c$ *is non-abelian. Then* $D$ *is biholomorphic to a domain of the form*

$$\mathcal{D}_3 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_{2m}(z_1, \overline{z}_1) < 0\}$$

*where* $P_{2m}(z_1, \overline{z}_1)$ *is a homogeneous polynomial of degree* $2m$ *without harmonic terms.*

*Proof.* We will again work with $G$ instead of $D$ as in the previous proposition. The first step is to show that the one-parameter subgroup $(T_t)$ is normal in $\mathrm{Aut}(G)^c$. Suppose not. The real vector field that generates $(T_t)$ is

$$X = i/2\ \partial/\partial z_2 - i/2\ \partial/\partial \overline{z}_2$$

and it is possible to find another real vector field $Y$ such that $X, Y$ generate $\mathfrak{g}(G)$ the Lie algebra of $\mathrm{Aut}(G)^c$ and

$$[X, Y] = \mu Y \tag{3.1}$$

for some non-zero real $\mu$, i.e., the one-parameter subgroup generated by $Y$ is normal in $\mathrm{Aut}(G)^c$. Let

$$Y = \Re\Big(f_1(z_1, z_2)\ \partial/\partial z_1 + f_2(z_1, z_2)\ \partial/\partial z_2\Big)$$

for some $f_1(z_1, z_2), f_2(z_1, z_2) \in \mathcal{O}(G)$. Now (3.1) is equivalent to

$$\partial f_1/\partial z_2 = -2i\mu f_1 \text{ and } \partial f_2/\partial z_2 = -2i\mu f_2$$

which yield

$$f_1(z_1, z_2) = \exp(-2i\mu z_2)\, h_1(z_1) \text{ and } f_2(z_1, z_2) = \exp(-2i\mu z_2)\, h_2(z_1).$$

Both $h_1(z_1)$ and $h_2(z_1)$ are entire functions since $G$ projects surjectively onto the $z_1$-axis. The special form of $f_1(z_1, z_2), f_2(z_1, z_2)$ shows that $Y$ generates a one-parameter subgroup of $\mathrm{Aut}(\mathbf{C}^2)$. Now suppose that $h_1(z_0) = 0$ for some $z_0 \in \mathbf{C}$. Then the restrictions of both $X, Y$ to the line $L = \{(z_1, z_2) \in \mathbf{C}^2 : z_1 = z_0\}$ are multiples of $\partial/\partial z_2$ and hence $\mathrm{Aut}(G)$ acts on $L$ and this action leaves the half-plane

$$G \cap L = \{(z_0, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P(z_0, \overline{z}_0) < 0\}$$

invariant. Hence the restriction of $\mathrm{Aut}(G)^c$ to $L$ can be identified with a two dimensional subgroup of $\mathrm{Aut}(\mathbf{C})$ that preserves a half-plane. Such a subgroup clearly contains the restriction of $(T_t)$ to $L$ as a normal subgroup and this contradicts the assumption that $(T_t)$ is not normal in $\mathrm{Aut}(G)^c$. The conclusion is that $h_1(z_1)$ is a non-vanishing entire function.

Now $Y$ is well defined on $\mathbf{C}^2$ and so the one-parameter subgroup generated by it maps the boundary of $G$ to itself. This implies that $Y$ is a tangential vector field, i.e.,

$$Y(2\Re z_2 + P(z_1, \overline{z}_1)) = 0$$

whenever $2\Re z_2 = -P(z_1, \overline{z}_1)$. This simplifies as

$$\Big(h_1(z_1)\, \partial P/\partial z_1 + h_2(z_1)\Big) + \exp(-2i\mu P(z_1, \overline{z}_1))\Big(\overline{h_1(z_1)}\, \partial P/\partial \overline{z}_1 + \overline{h_2(z_1)}\Big) = 0$$

for all $z_1 \in \mathbf{C}$.

*Claim:* Define $F(z, \overline{z}) = \Big(h_1(z)\, \partial P/\partial z + h_2(z)\Big) + \exp(-2i\mu P(z, \overline{z}))\Big(\overline{h_1(z)}\, \partial P/\partial \overline{z} + \overline{h_2(z)}\Big)$ a real analytic, complex valued function on the plane. If $F(z, \overline{z}) \equiv 0$ for all $z \in \mathbf{C}$, then $h_1(z)$ must vanish somewhere.

To see this let

$$h_1(z) = \sum_{j \geq 0} c_j z^j \text{ and } h_2(z) = \sum_{j \geq 0} d_j z^j$$

where $c_0 = h_1(0) \neq 0$ by assumption. Also write

$$P(z, \overline{z}) = \overline{z}^r q_r(z) + \overline{z}^{r+1} q_{r+1}(z) + \ldots$$

where the sum is finite and the coefficients $q_r(z), q_{r+1}(z), \ldots$ are holomorphic polynomials. Note that $r \geq 1$ since $P(z, \overline{z})$ does not have harmonic terms. For the same reason none of $q_r(z), q_{r+1}(z), \ldots$ have a constant term nor are any of them identically equal to a constant. Suppose that $r > 1$. Then the coefficient of $\overline{z}^{r-1}$ in

$$\begin{aligned} F(z, \overline{z}) = &\Big((c_0 + c_1 z + \ldots)(\overline{z}^r q_r'(z) + \overline{z}^{r+1} q_{r+1}'(z) + \ldots) + (d_0 + d_1 z + \ldots)\Big) \\ &+ \Big(1 - 2i\mu(\overline{z}^r q_r(z) + \overline{z}^{r+1} q_{r+1}(z) + \ldots) + \ldots\Big)\Big((\overline{c}_0 + \overline{c}_1 \overline{z} + \ldots)(r\overline{z}^{r-1} q_r(z) + (r+1)\overline{z}^r q_{r+1}(z) \\ &+ \ldots) + (\overline{d}_0 + \overline{d}_1 \overline{z} + \ldots)\Big) \end{aligned}$$

is $\overline{d}_{r-1} + \overline{c}_0 r q_r(z)$ and this must be identically zero. Thus $q_r(z)$ is a constant which is not possible. It follows that $r = 1$. In this case the holomorphic part of the above expansion, which must also be identically zero, can be shown to be $(h_2(z) + \overline{c}_0 q_1(z) + \overline{d}_0)$. Hence

$$h_2(z) = -(\overline{c}_0 q_1(z) + \overline{d}_0) \tag{3.2}$$

which implies that $h_2(z)$ is a polynomial. The coefficient of $\overline{z}$, which must also be identically zero, is $(h_1(z) q_1'(z) + \overline{c}_1 q_1(z) + 2\overline{c}_0 q_2(z) + \overline{d}_1 - 2i\mu(\overline{c}_0 q_1(z) + \overline{d}_0))$. Hence

$$h_1(z) = \Big(2i\mu(\overline{c}_0 q_1(z) + \overline{d}_0) - (\overline{c}_1 q_1(z) + 2\overline{c}_0 q_2(z) + \overline{d}_1)\Big)\Big/ q_1'(z) \tag{3.3}$$

which implies that $h_1(z)$ is a rational function. If the degree of this rational function is atleast one then $h_1(z)$ will vanish somewhere. The only other possibility is that the rational function is a constant which forces $h_1(z) \equiv c_0$.

In case $h_1(z) \equiv c_0$, consider the new holomorphic function obtained by complexifying $F(z, \overline{z})$, i.e., by replacing $\overline{z}$ by an independent complex variable $w$ we can consider

$$\tilde{F}(z,w) = \Big(c_0\ \partial P/\partial z\ (z,w) + h_2(z)\Big) + \exp(-2i\mu P(z,w)\Big(\overline{c}_0\ \partial P/\partial w\ (z,w) + \overline{h}_2(w)\Big) \in \mathcal{O}(\mathbf{C}^2)$$

where $\overline{h}_2(w) = \overline{h_2(\overline{w})}$. Note that $\tilde{F}(z,w)$ vanishes when $w = \overline{z}$ and so $\tilde{F}(z,w) \equiv 0$. This shows that

$$\exp(-2i\mu P(z,w)) \equiv -\Big(c_0\ \partial P/\partial z\ (z,w) + h_2(z)\Big)\Big/\Big(\overline{c}_0\ \partial P/\partial w\ (z,w) + \overline{h}_2(w)\Big). \tag{3.4}$$

The left side above is the exponential of a non-constant polynomial $P(z,w)$ and hence there is $\lambda \in \mathbf{C}^*$ such that the restriction of the left side to the line $L = \{w = \lambda z\}$ is non-constant. Moreover it is non-vanishing as well. However the right side in (3.4) is a rational function on $L$ and this must vanish somewhere. This is a contradiction and the claim follows.

Now $(T_t)$ is normal in $\mathrm{Aut}(G)^c$ and let $(S_s)$ be a one-parameter subgroup that generates $\mathrm{Aut}(G)^c$ along with $(T_t)$. Then for all real $s, t$

$$S_s \circ T_t = T_{\theta(s,t)} \circ S_s \tag{3.5}$$

for some smooth function $\theta(s,t)$. Composing with $T_t$ once more on the right gives

$$S_s \circ T_{2t} = T_{\theta(s,t)} \circ S_s \circ T_t = T_{2\theta(s,t)} \circ S_s$$

which shows that $\theta(s, 2t) = 2\theta(s,t)$ and inductively we get $\theta(s, mt) = m\theta(s,t)$ for all integers $m \geq 1$. Putting $t = 0$ it follows that $\theta(s, 0) = 0$ for all $s$. Now for $t_1, t_2, s \in \mathbf{R}$, (3.5) shows that

$$S_s \circ T_{t_1+t_2} = S_s \circ T_{t_1} \circ T_{t_2} = T_{\theta(s,t_1)+\theta(s,t_2)} \circ S_s$$

which gives $\theta(s, t_1 + t_2) = \theta(s, t_1) + \theta(s, t_2)$ and hence that $\theta(s,t) = tf(s)$ for some smooth function $f(s)$. Again for $s_1, s_2, t \in \mathbf{R}$, (3.5) shows that

$$S_{s_1+s_2} \circ T_t = S_{s_1} \circ S_{s_2} \circ T_t = T_{\theta(s_1,\theta(s_2,t))} \circ S_{s_1}$$

which gives $\theta(s_1 + s_2, t) = \theta(s_1, \theta(s_2, t))$. In terms of $f(s)$ this means that $f(s_1 + s_2) = f(s_1)f(s_2)$. Hence $f(s) = \exp(\alpha s)$ for some non-zero real $\alpha$. If $S_s(z_1, z_2) = (S_s^1(z_1, z_2), S_s^2(z_1, z_2))$ then (3.5) is equivalent to

$$S_s^1(z_1, z_2 + it) = S_s^1(z_1, z_2), \text{and}$$
$$S_s^2(z_1, z_2 + it) = S_s^2(z_1, z_2) + i\exp(\alpha s)t$$

for all real $s, t$. As in proposition 3.1, the first equation forces $S_s^1(z_1, z_2) = S_s^1(z_1)$ and that $(S_s^1(z_1))$ is a non-trivial subgroup of $\mathrm{Aut}(\mathbf{C})$. The second equation implies that $\partial S_s^2/\partial z_2 = \exp(\alpha s)$, i.e., $S_s^2(z_1, z_2) = \exp(\alpha s)z_2 + f(s, z_1)$ where $z_1 \mapsto f(s, z_1)$ is entire for all $s \in \mathbf{R}$. It follows that $(S_s)$ is a one-parameter subgroup of $\mathrm{Aut}(\mathbf{C}^2)$ that preserves $G$. An argument similar to that in proposition 3.1 shows that $f(s, z_1)$ is a holomorphic polynomial for all $s$. Moreover after a change of coordinates in $z_1$, it follows that $S_s^1(z_1) = z_1 + is$ or $S_s^1(z_1) = \exp(\beta s)z_1$ for some $\beta \in \mathbf{C} \setminus \{0\}$.

If $S_s(z_1, z_2) = (z_1 + is, \exp(\alpha s)z_2 + f(s, z_1))$ then it follows that

$$P(z_1, \overline{z}_1) = 2\exp(-\alpha s)(\Re f(s, z_1)) + \exp(-\alpha s)P(z_1 + is, \overline{z}_1 - is)$$

for all $z_1 \in \mathbf{C}$ and $s \in \mathbf{R}$. This cannot hold by considering $P_N(z_1, \overline{z}_1)$ the homogeneous summand of highest degree in $P(z_1, \overline{z}_1)$. On the other hand, if $S_s(z_1, z_2) = (\exp(\beta s)z_1, \exp(\alpha s)z_2 + f(s, z_1))$ it follows that

$$P(z_1, \overline{z}_1) = 2\exp(-\alpha s)(\Re f(s, z_1)) + \exp(\alpha s)P(\exp(\beta s)z_1, \exp(\overline{\beta}s)\overline{z}_1)$$

for all $z_1 \in \mathbf{C}$ and $s \in \mathbf{R}$. But $P(z_1, \overline{z}_1)$ has no harmonic summands and this forces

$$P(z_1, \overline{z}_1) = \exp(-\alpha s)P(\exp(\beta s)z_1, \exp(\overline{\beta}s)\overline{z}_1).$$

Since $P(z_1, \overline{z}_1)$ is not identically zero there is at least one non-vanishing homogeneous summand in it. Let the degree of this summand be $k > 0$. By comparing terms of degree $k$ on both sides it follows that $\alpha = k(\Re\beta)$. This shows that $P(z_1, \overline{z}_1) = P_k(z_1, \overline{z}_1)$ is homogeneous of degree $k$.

If $k$ is odd it is known that the envelope of holomorphy of the model domain

$$G = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_k(z_1, \overline{z}_1) < 0\}$$

contains a full open neighbourhood of the origin in $\mathbf{C}^2$. Let $f : D \to G$ be a biholomorphism. Then $f^{-1}$ extends holomorphically across $0 \in \partial G$ and it follows that $f^{-1}(0) \in \partial D$ belongs to $\hat{D}$. Now note that the dilations $S_s(z_1, z_2) = (\exp(s/k)z_1, \exp(s)z_2),\ s \in \mathbf{R}$ are automorphisms of $G$ that cluster at the origin and thus the orbit of $f^{-1}(z_1, z_2) \in D$ under the one-parameter group $f^{-1} \circ S_s \circ f \in \mathrm{Aut}(D)$ clusters at $f^{-1}(0) \in \hat{D}$. Since $D$ is bounded, this contradicts the Greene-Krantz observation mentioned earlier. Hence $k = 2m$ is even. $\square$

## 4. Model domains when Aut($D$) is three dimensional

Consider the natural action of $\mathrm{Aut}(D)^c$ on $D$. It was shown in [23] that for every $p \in D$, the $\mathrm{Aut}(D)^c$-orbit $O(p)$, which is a connected closed submanifold in $D$, has real codimension 1 or 2. In case $O(p)$ has codimension 2, then it is either a complex curve biholomorphically equivalent to $\Delta$, the unit disc in the plane or else maximally totally real in $D$. In case the codimension is 1, then $O(p)$ is either a strongly pseudoconvex hypersurface or else Levi flat everywhere. In the latter case the leaves of the Levi foliation are biholomorphically equivalent to $\Delta$ and the explicit analysis in proposition 4.1 of [23] shows that each leaf is closed in $D$. Moreover it was also shown that if there are no codimension 1 orbits then $D \backsimeq \Delta \times R$ where $R$ is any hyperbolic Riemann surface with a discrete group of automorphisms. In our case $R$ is forced to be simply connected since $D$ is so and hence $R \backsimeq \Delta$. This is a contradiction. Thus there is at least one codimension 1 orbit in $D$.

The main goal in this section will be to identify which of the examples that occur in [23] can be equivalent to $D$ as in the main theorem. It will turn out that $D \backsimeq \mathcal{D}_4 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + (\Re z_1)^{2m} < 0\} \backsimeq R_{1/2m,-1,1}$, which is defined below. For this purpose the classification in [23] is divided into groups, the examples in each group having a distinguishing property. Whether $D$ can sustain such properties is then checked case by case. Only the relevant properties of the examples have been listed for ready reference in the following subsections. The reader is referred to [23] for more detailed information and it must be pointed out that the notation used to describe the domains below is the same as in [23]. Moreover unless stated otherwise, the word 'orbit' in what follows will always refer to the $\mathrm{Aut}(D)^c$-orbit.

4.1. **Some examples with Levi flat orbits:** The following domains admit Levi flat orbits:

- Fix $b \in \mathbf{R}$, $b \neq 0, 1$ and let $-\infty \leq s < 0 < t \leq \infty$ where at least one of $s, t$ is finite. Define

$$R_{b,s,t} = \Big\{(z_1, z_2) \in \mathbf{C}^2 : s(\Re z_1)^b < \Re z_2 < t(\Re z_1)^b,\ \Re z_1 > 0\Big\}.$$

which has a unique Levi flat orbit given by

$$\mathcal{O}_1 = \Big\{(z_1, z_2) \in \mathbf{C}^2 : \Re z_1 > 0,\ \Re z_2 = 0\Big\}.$$

Note that $R_{1/2,s,-s} \backsimeq \mathbf{B}^2$ for all $s < 0$ and therefore these values of the parameters $b, s, t$ will not be considered.

- For $b > 0, b \neq 1$ and $-\infty < s < 0 < t < \infty$ define

$$\hat{R}_{b,s,t} = R_{b,s,\infty} \cup \Big\{(z_1, z_2) \in \mathbf{C}^2 : \Re z_2 > t(-\Re z_1)^b,\ \Re z_1 < 0\Big\} \cup \hat{\mathcal{O}}_1$$

where

$$R_{b,s,\infty} = \Big\{(z_1, z_2) \in \mathbf{C}^2 : s(\Re z_1)^b < \Re z_2,\ \Re z_1 > 0\Big\}$$

and

$$\hat{\mathcal{O}}_1 = \Big\{(z_1, z_2) \in \mathbf{C}^2 : \Re z_1 = 0,\ \Re z_2 > 0\Big\}.$$

- For $-\infty < t < 0 < s < \infty$ define

$$\hat{U}_{s,t} = U_{s,\infty} \cup \Big\{(z_1, z_2) \in \mathbf{C}^2 : \Re z_1 > \Re z_2 \cdot \ln(t\, \Re z_2),\ \Re z_2 < 0\Big\} \cup \mathcal{O}_1$$

where

$$U_{s,\infty} = \Big\{(z_1, z_2) \in \mathbf{C}^2 : \Re z_2 \cdot \ln(s\, \Re z_2) < \Re z_1,\ \Re z_2 > 0\Big\}$$

and $\mathcal{O}_1$ is as in the definition of $R_{b,s,t}$ above.

**Proposition 4.1.** *Let $D$ be as in the main theorem. If $D$ admits a Levi flat orbit then it is equivalent to $\mathcal{D}_4 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + (\Re z_1)^{2m} < 0\}$.*

Recall that $D$ is not pseudoconvex near $p_\infty$. By [6] this is equivalent to saying that none of the orbits in $D$ accumulate at the weakly pseudoconvex points near $p_\infty$. The proof depends on several intermediate steps. First assume that the orbit accumulation point $p_\infty \in T_1$. Choose coordinates around $p_\infty = 0$ so that $T_1$ coincides with the imaginary $z_1$-axis near the origin and fix a polydisk $U = \{|z_1| < \eta, |z_2| < \eta\}$ centered at the origin with $\eta > 0$ small enough so that $U \cap T_1$ is an embedded piece of the $y_1$-axis. The function $\tau(z) = (\Re z_1)^2 + |z_2|^2$ is then a non-negative, strongly plurisubharmonic function in $U$ whose zero locus is exactly $U \cap T_1$. Also note that $i\partial\overline{\partial}\tau = i\partial\overline{\partial}|z|^2$. For $s > 0$ and $E \subset \mathbf{C}^2$ let $\mathcal{H}_s(E)$ denote the $s$-dimensional Hausdorff measure of $E$. Finally for a sequence of closed sets $E_j$ in some domain $W \subset \mathbf{C}^2$, define their cluster set $\mathrm{cl}(E_j)$ as

$$\mathrm{cl}(E_j) = \{z \in W : \text{there exists } z_j \in E_j \text{ such that } z \text{ is an accummulation point of the sequence } \{z_j\}\}.$$

The relevant case will be the one in which the $\{E_j\}$ are complex analytic sets in $W$ of a pure fixed dimension. In this situation the following theorem that was proved by Diederich-Pinchuk in [15] will be useful.

*Let $M \subset W \subset \mathbf{C}^2$ be a smooth real analytic hypersurface of finite type. Let $E_j \subset W$ be closed complex analytic sets of pure dimension 1. Then $\mathrm{cl}(E_j)$ is not contained in $M$.*

**Proposition 4.2.** *Let $A_j \subset U \cap D$ be a sequence of pure one dimensional complex analytic sets such that $\partial A_j \subset \partial U \cap D$ for all $j$. Assume that there are points $a_j \in A_j$ such that $a_j \to a \in U \cap T_1$. Then there exists a compact set $K \subset U$ disjoint from $T_1$ such that $A_j \cap K \neq \emptyset$ for all large $j$.*

*Proof.* Observe that each $A_j \subset U$ is analytic since none of them clusters at points of $U \cap \partial D$. The domain $\Omega_r = \{(z \in U : \tau(z) < r\}$ is a strongly pseudoconvex tubular neighbourhood of $U \cap T_1$. Let $0 < r \ll \eta$ whose precise value will be fixed later. Then only finitely many $A_j$ can be contained in $\Omega_r$. Indeed if this does not hold then there is a sequence, still to be denoted by $A_j$, for which each $A_j \subset \Omega_r$. Define $\varrho_j(z) = \tau(z) - |z - a_j|^2/2$ and note that $\varrho_j(a_j) = \tau(a_j) > 0$. Moreover

$$i\partial\overline{\partial}\varrho_j = i\partial\overline{\partial}\tau - i\partial\overline{\partial}|z - a_j|^2/2 = i\partial\overline{\partial}\tau - i\partial\overline{\partial}|z|^2/2 = i\partial\overline{\partial}|z|^2/2 > 0$$

shows that the restriction of $\varrho_j$ to $A_j$ is subharmonic for all $j$. Now fix $j$ and let $w \in \partial A_j \subset \partial U \cap D$. Then

$$\varrho_j(w) = \tau(w) - |w - a_j|^2/2 \leq r - |w - a_j|^2/2 < 0$$

the last inequality holding whenever $r > 0$ is chosen to satisfy

$$2r < (\eta - |a|)^2 \approx (\eta - |a_j|)^2 \leq (|z| - |a_j|)^2 \leq |z - a_j|^2$$

for all $z \in \partial U$. This contradicts the maximum principle and hence all except finitely many $A_j$ must intersect $U \cap \{\tau(z) \geq r\}$. □

*Remark:* The above lemma is a version for sequences of analytic sets of the well known fact (see [9] for example) that an analytic set of positive dimension cannot approach a totally real manifold tangentially. Moreover the choice of $r$ depended upon the distance of the limit point $a$ from $\partial U$. It is clear from the proof that a uniform $r$ can be chosen if $a$ is allowed to vary in a relatively compact subset of $U \cap T_1$. Finally, the theorem on the cluster set of analytic sets mentioned above could have been used at this stage to conclude that $\mathrm{cl}(A_j)$ intersects $U^-$. However a more precise description of the sub-domain in $U^-$ that intersects $\mathrm{cl}(A_j)$ is afforded by the above lemma and this will be needed in the sequel.

**Lemma 4.3.** *Let $q \in D$ be such that $O(q) \subset D$ is Levi flat. Fix a ball $B_w(r)$ around some $w \in \partial D$ for some $r > 0$ and suppose that $O(q) \cap B_w(r) \neq \emptyset$. Let $C_q \subset O(q) \cap B_w(r)$ be a connected component. Then $C_q$ is itself a closed Levi flat hypersurface that admits a codimension one foliation by closed hyperbolic Riemann surfaces each of which is itself closed in $B_w(r) \cap D$. Suppose further that there is a leaf $R$ in $C_q$ that does not cluster at points of $B_w(r) \cap \partial D$. Then there is an arbitrarily small neighbourhood $W$ of $R$, $W$ open in $\mathbf{C}^2$ and compactly contained in $D$ with the property that if $S \subset C_q$ is leaf with $S \cap W \neq \emptyset$ then $S \subset W$.*

*Proof.* $O(q)$ has all the above mentioned properties and hence $C_q$ inherits them as well. Now choose an arbitrary neighbourhood $\tilde{W}$ of $R$, $\tilde{W}$ open in $\mathbf{C}^2$ and compactly contained in $D$. If needed shrink it so that $C_q \cap \tilde{W}$ is connected. Note that the leaves of $C_q \cap \tilde{W}$ are exactly the CR orbits on $C_q \cap \tilde{W}$. Choose a real one dimensional section $T \subset C_q \cap \tilde{W}$ that passes through some $r \in R$ and which is transverse to the leaves of $C_q \cap \tilde{W}$ near $r$. Since the foliation of $C_q$ exists in a neighbourhood of the closure of $\tilde{W}$, it follows that if $T$ is small enough, the CR orbits of $C_q \cap \tilde{W}$ through points in $T$ stay close to $R$. The existence of $W \subset \tilde{W}$ follows. □

**Proposition 4.4.** *Suppose there exists $q \in D$ such that $\overline{O(q)} \cap (U \cap T_1) \neq \emptyset$. Then it is not possible to find a component of $O(q) \cap U$, say $C_q$ such that $\overline{C}_q \cap (U \cap T_1) \neq \emptyset$. In particular the number of components of $O(q) \cap U$ is not finite.*

*Proof.* First observe that $O(q)$ cannot cluster at points of $(U \cap \partial D) \setminus T_1$ for these contain pseudoconvex or pseudoconcave points and while all model domains are known in the former case, the latter possibility is ruled out as these points belong to $\hat{D}$. The hypotheses therefore imply that $\overline{O(q)} \cap U \subset T_1$. To argue by contradiction, suppose that $C_q \subset O(q) \cap U$ is a component that satisfies $\overline{C}_q \cap (U \cap T_1) \neq \emptyset$. Pick $a_j \in C_q$ such that $a_j \to a \in U \cap T_1$. Two cases now arise:

*Case 1:* None of the leaves of $C_q$ clusters at points of $U \cap T_1$.

Let $R_j$ be the leaves of $C_q$ that contain $a_j$ for $j \geq 1$. It follows from proposition 4.2 that each $R_j$ (except finitely many which can be ignored) must intersect $U^- \cap \{\tau(z) \geq r\}$ for a fixed small $r > 0$. For $0 < \delta \ll r$, let

$$U_\delta^- = \{z \in U^- : -\delta < \rho(z) < 0\}$$

where $\rho(z)$ is a defining function for $U \cap \partial D$. $U_\delta^-$ is then a one-sided collar around $U \cap \partial D$ of width $\delta$. Since $C_q$ cannot cluster at points of $(U \cap \partial D) \setminus T_1$ it follows that each $R_j$ must intersect

$$K_{r,\delta}^- = (U^- \cap \{\tau(z) \geq r\}) \setminus U_\delta^- \tag{4.1}$$

which is compact in $D$. Choose $\alpha_j \in R_j \cap K_{r,\delta}^-$ and let $\alpha_j \to \alpha \in K_{r,\delta}^-$ after perhaps passing to a subsequence. Since $C_q$ is closed in $U \setminus T_1$ it follows that $\alpha \in C_q$. Let $R_\alpha$ be the leaf of $C_q$ that contains $\alpha$. By the hypothesis of case 1, $R_\alpha$ does not cluster at $U \cap T_1$ and hence it stays uniformly away from $U \cap \partial D$. By lemma 4.3 there is a neighbourhood $W$ of $R_\alpha$ that is compactly contained in $D$ such that $R_j \subset W$ for all large $j$. But then $a_j \in R_j \subset W$ and this contradicts the fact that $a_j \to a \in U \cap T_1$.

*Case 2:* There is at least one leaf $R \subset C_q$ that clusters at points of $U \cap T_1$.

There are two subcases to consider. First if $\mathcal{H}_1(\overline{R} \cap U \cap T_1) = 0$ it follows from Shiffman's theorem that $\overline{R} \subset U$ is a closed, one dimensional complex analytic set. But then $\overline{R} \subset \overline{U^-}$ and so the strong disk theorem shows that all points in $\overline{R} \cap U \cap T_1$ lie in the envelope of holomorphy of $D$. This is a contradiction. Second if $\mathcal{H}_1(\overline{R} \cap U \cap T_1) > 0$ then $R$ admits analytic continuation across $T_1$, i.e., there is a neighbourhood $V$ of $U \cap T_1$ and a closed complex analytic set $A \subset V$ of pure dimension one such that $R \cap V \subset A$. In fact the uniqueness theorem shows that $A = T_1^{\mathbf{C}}$ the complexification of $U \cap T_1$ and $R \cup T_1^{\mathbf{C}}$ is analytic in $U^- \cup V$. In particular $U \cap T_1 \subset \overline{R}$.

Let $R'$ be a leaf in $C_q$ that is distinct from $R$. Then $R'$ cannot cluster at $U \cap T_1$. Indeed if $\mathcal{H}_1(\overline{R}' \cap U \cap T_1) = 0$ then as before all points in $\overline{R}' \cap U \cap T_1$ will be in the envelope of holomorphy of $D$ which cannot happen or else both $R'$ and $R$ have the same analytic continuation, namely $T_1^{\mathbf{C}}$. The uniqueness theorem shows that $R' = R$ which is a contradiction. The conclusion is that no leaf apart from $R$ can cluster at $U \cap T_1$.

By the remark before lemma 4.3 it is possible to choose $r > 0$ and $0 < \delta \ll r$ such that $R \cap K_{r,\delta}^- \neq \emptyset$ where $K_{r,\delta}^-$ is as in (4.1). Fix $\alpha \in R \cap K_{r,\delta}^-$ and choose $\alpha_j \in C_q \cap K_{r,\delta}^-$ such that $\alpha_j \to \alpha$. Let $R_j$ be the leaves of $C_q$ that contain $\alpha_j$. Choose $\beta_j \in R$ such that $\beta_j \to \beta \in U \cap T_1$. By the proof of lemma 4.3 it is possible to find a subsequence, still denoted by $R_j$, and points $r_j \in R_j$ such that $|r_j - \beta_j| \to 0$. This implies that $r_j \to \beta$. Since $R_j \cap K_{r,\delta}^- \neq \emptyset$ for all $j$, it follows that their cluster set $\mathrm{cl}(R_j) \neq \emptyset$ in $U^-$. Choose $c \in \mathrm{cl}(R_j) \cap K_{r,\delta}^-$ and let $R_c$ be the leaf of $C_q$ that contains it. If $R_c$ does not cluster at $U \cap \partial D$ then lemma 4.3 shows that $R_j$ are compactly contained in $D$ contradicting the fact that $r_j \in R_j$ is such

that $r_j \to \beta$. Hence $R_c$ must cluster at $U \cap T_1$ and therefore $R_c = R$. This shows that the cluster set $\mathrm{cl}(R_j)$ in $U^-$ is exactly $R$.

To conclude translate coordinates so that $\beta = 0$ and let $L$ be a complex line that is tangent to $T_1^{\mathbf{C}}$ at the origin. Since $R$ coincides with $T_1^{\mathbf{C}}$ near the origin it follows that $L$ must intersect $U^-$. Choose a small polydisk $U_1 \times U_2$ around the origin such that $U_1$ is parallel to $L$. The projection

$$\pi : T_1^{\mathbf{C}} \cap (U_1 \times U_2) \to U_1$$

is then proper for an appropriate choice of $U_1, U_2$. This is equivalent to the condition that $T_1^{\mathbf{C}}$ has no limit points on $U_1 \times \partial U_2$. Now $R_j \cap (U_1 \times U_2)$ are pure one dimensional analytic sets in $U^- \cap (U_1 \times U_2)$ that do not cluster at $U \cap \partial D$. Hence they are analytic in $U_1 \times U_2$. Moreover the cluster set of $\{R_j \cap (U_1 \times U_2)\}$ in $U^-$ is exactly $R \cap U^- \cap (U_1 \times U_2) = T_1^{\mathbf{C}} \cap U^- \cap (U_1 \times U_2)$. There can be no other points in $(U \cap \partial D) \setminus T_1$ that lie in $\mathrm{cl}(R_j \cap (U_1 \times U_2))$ as the Diederich-Pinchuk theorem mentioned above shows and hence $R_j \cap (U_1 \times U_2)$ has no limit points on $U_1 \times \partial U_2$ for all large $j$, i.e., the projection

$$\pi : R_j \cap (U_1 \times U_2) \to U_1$$

is proper for all large $j$. Therefore $\pi(R_j \cap (U_1 \times U_2)) = U_1$ and this forces $R_j$ to cluster at points of $U \cap \partial D$ as the disk $U_1$ intersects both $U^{\pm}$. This is a contradiction. $\square$

*Proof of Proposition 4.1*: Suppose $q \in D$ is such that $O(q)$ is Levi flat. By lemma 2.2 it is known that $\phi_j(q) \to p_\infty = 0$ after a translation of coordinates. Two cases need to be considered.

*Case 1:* For large $j$ none of the $\phi_j$'s belong to $\mathrm{Aut}(D)^c$.

Since $\mathrm{Aut}(D)^c$ is normal in $\mathrm{Aut}(D)$ it follows that $\phi_j(O(q)) = O(\phi_j(q))$ for all $j$. Thus $O(\phi_j(q))$ is a family of distinct Levi flat hypersurfaces for all large $j$. Fix an arbitrarily small neighbourhood $U$ of the origin. Then observe that $O(\phi_j(q)) \cap U^- \neq \emptyset$ for $j$ large and let $C_j \subset O(\phi_j(q)) \cap U^-$ be the connected components that contain $\phi_j(q)$. The proof of proposition 4.4 shows that none of the $C_j$'s clusters at $U \cap T_1$. Let $R_j \subset C_j$ be the leaves such that $\phi_j(q) \in R_j$. Then $R_j \subset U$ are pure one dimensional analytic sets and it follows from the Diederich-Pinchuk result and proposition 4.2 that $\mathrm{cl}(R_j) \cap U^- \cap \{\tau(z) \geq r\} \neq \emptyset$ for a suitable $r > 0$. In fact more can be said about $\mathrm{cl}(R_j)$; indeed let $a \in \mathrm{cl}(R_j) \cap U^-$ and pick $a_j \in R_j$ such that $a_j \to a$ after perhaps passing to a subsequence. Choose $f \in \mathrm{Aut}(D)^c$ such that $f_j \circ \phi_j(q) = a_j$. Then $\{f_j \circ \phi_j\} \to f \in \mathrm{Aut}(D)$ since $f_j \circ \phi_j(q) = a_j \to a \in U^-$. It follows that $\mathrm{cl}(R_j) \cap U^-$ is contained in the $\mathrm{Aut}(D)$-orbit of $q$.

This observation now shows that $\mathrm{cl}(R_j) \cap ((U \cap \partial D) \setminus T_1) = \emptyset$ as otherwise the orbit of $q$ will cluster at either pseudoconvex or pseudoconcave points. All model domains are known in the former case while the latter possibility does not hold. Therefore it is possible to find $r > 0$ and $0 < \delta \ll r$ such that $\mathrm{cl}(R_j) \cap K^-_{r,\delta} \neq \emptyset$ where $K^-_{r,\delta}$ is as in (4.1). Choose $c_j \in R_j \cap K^-_{r,\delta}$ such that $c_j \to c_0 \in \mathrm{cl}(R_j) \cap K^-_{r,\delta}$ after passing to a subsequence and re-indexing and let $d_j \in R_j$ converge to $d_0 \in U^-$. Let $g_j \in \mathrm{Aut}(D)^c$ be such that $g_j(c_j) = d_j$. Let $g$ be the holomorphic limit of the $g_j$'s and observe that $g(c_0) = d_0$. Therefore $g \in \mathrm{Aut}(D)^c$, this being closed in $\mathrm{Aut}(D)$. This shows that $\mathrm{cl}(R_j) \cap U^- \subset O(c_0) \cap U^-$. This strengthens the observation made above that $\mathrm{cl}(R_j) \cap U^-$ is contained in the $\mathrm{Aut}(D)$-orbit of $q$. Moreover since the analytic sets $\{R_j\}$ contain points arbitrarily close to the origin it follows that $O(c_0)$ also clusters there.

To conclude, note that there are infinitely many distinct components of $O(c_0) \cap U^-$ by proposition 4.4. If $a_j \in O(c_0) \cap U^-$ converges to the origin, we can consider the distinct components $S_j \subset O(c_0) \cap U^-$ that contain $a_j$ and the one dimensional analytic sets $A_j \subset S_j$ that contain $a_j$. By proposition 4.2 all except finitely many $A_j$ (and hence the same for $S_j$) intersect $K^-_{r,\delta}$ for some $r > 0$ and $0 < \delta \ll r$. This argument shows that there are infinitely many components of $O(c_0) \cap U^-$ that intersect a compact subset of $D$ and this forces $O(c_0)$ to cluster on itself in $D$. This is a contradiction since $O(c_0)$ is a closed submanifold of $D$.

*Case 2:* After passing to a subsequence and re-indexing, all the $\phi_j$'s are contained in $\mathrm{Aut}(D)^c$.

In this case it can be seen that $O(q)$ itself clusters at the origin. By proposition 4.4 there are infinitely many distinct components of $O(q) \cap U^-$ and now the arguments in the last paragraph of case 1 apply to provide a contradiction. In particular the intermediate step about finding $c_0 \in K^-_{r,\delta}$ is not needed.

The steps leading up to this point were all proved under the assumption that $p_\infty \in T_1$. In case $p_\infty \in T_0$, let $\gamma \subset U \cap \partial D$ be a germ of an embedded real analytic arc that contains $p_\infty$ in its interior and which intersects any stratum in $T_1$ clustering at $p_\infty$ in a discrete set only. Such a choice is possible by the local finiteness of the semi-analytic stratification of $\mathcal{B}$, the border between the pseudoconvex and pseudoconcave points. The complement of $\gamma$ in $U \cap \partial D$ near $p_\infty$ consists of pseudoconvex or pseudoconcave points or those that lie on $T_1$. The arguments mentioned above show that if there is a Levi flat orbit in $D$ then it cannot cluster at points on $T_1$. The same steps now can be applied with $T_1$ replaced by $\gamma$ and this finally shows that if $D$ is not pseudoconvex then it cannot admit Levi flat orbits. It should be noted that the arguments used to obtain this conclusion did not use the fact that $\dim \mathrm{Aut}(D) = 3$. Only the existence of a Levi flat orbit with closed leaves each of which is equivalent to the unit disc was used.

To continue, if there is one such orbit then $D$ must be a pseudoconvex domain, i.e., the boundary $\partial D$ near $p_\infty$ is weakly pseudoconvex and of finite type. It follows by [6] that $D \backsimeq \tilde{D} = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_{2m}(z_1, \overline{z}_1) < 0\}$ where $P_{2m}(z_1, \overline{z}_1)$ is a homogeneous subharmonic polynomial of degree $2m$ without harmonic terms. Now observe that $\tilde{D}$ is invariant under the one-parameter subgroups $T_t(z_1, z_2) = (z_1, z_2 + it)$ and $S_s(z_1, z_2) = (\exp(s/2m)z_1, \exp(s)z_2)$ and the corresponding real vector fields are $X = \Re(i\,\partial/\partial z_2)$ and $Y = \Re((z_1/2m)\,\partial/\partial z_1 + z_2\partial/\partial z_2)$. It can be seen that $[X, Y] = X$. Let $\mathfrak{h} \subset \mathfrak{g}(\tilde{D})$ be the Lie subalgebra generated by $X, Y$. Let $X, Y, Z$ be the generators of $\mathfrak{g}(\tilde{D})$. Using the Jacobi identity it can be seen that $\mathfrak{g}(\tilde{D})$ must be isomorphic to one of the following:

(a) $[X, Y] = X,\ [Z, X] = 0,\ [Z, Y] = \alpha Z$ for some real $\alpha$
(b) $[X, Y] = X,\ [Z, X] = 0,\ [Z, Y] = X + Z$
(c) $[X, Y] = X,\ [Z, X] = Y,\ [Z, Y] = -Z$

There are two possibilities that arise in case (a), namely when $\alpha = 0$ and when $\alpha \neq 0$. In the former case, lemma 5.3 of [36] can be applied to show that $\tilde{D} \backsimeq \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0\}$ and thus $\dim \mathrm{Aut}(\tilde{D}) = 4$ which is a contradiction. In the latter case, lemma 5.6 of [36] shows that $\tilde{D} \backsimeq \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + (\Re z_1)^{2m} < 0\}$ which is isomorphic to

$$R_{1/2m,-1,1} = \Big\{(z_1, z_2) \in \mathbf{C}^2 : -(\Re z_1)^{1/2m} < \Re z_2 < (\Re z_1)^{1/2m},\ \Re z_1 > 0\Big\}.$$

This domain has a unique Levi flat orbit namely $\mathcal{O}_1$. The structure of $\mathfrak{g}(\tilde{D})$ in case (b) is similar to that of case (a) when $\alpha \neq 0$. As above we have a contradiction. Finally in case (c) we have $\mathfrak{g}(\tilde{D}) \backsimeq \mathfrak{so}_{2,1}(\mathbf{R}) \backsimeq \mathfrak{sl}_2(\mathbf{R})$. A detailed calculation for this case has been done in lemma 5.8 in [36] (for this situation only the calculations done there are needed; none of the arguments that lead up to lemma 5.8 are needed here as the conditions are fulfilled in our case) which again shows that $\tilde{D} \backsimeq \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0\}$ and hence that $\dim \mathrm{Aut}(\tilde{D}) = 4$. This is not possible.

*Remark:* Proposition 4.1 shows that $D$ cannot be equivalent to either $\hat{R}_{b,s,t}$ or $\hat{U}_{s,t}$. A different class of examples with Levi flat orbits are also mentioned in [23], namely those that are obtained by considering finite and infinite sheeted covers of $D_{s,t}$ and $\Omega_{s,t}$. These will be defined later.

4.2. **Two tube domains:** Here we consider the following examples each of which is a tube domain over an unbounded base in the $(\Re z_1, \Re z_2)$ plane.

• For $0 \le s < t \le \infty$ where either $s > 0$ or $t < \infty$ define

$$U_{s,t} = \Big\{(z_1, z_2) \in \mathbf{C}^2 : \Re z_2 \cdot \ln(s\Re z_2) < \Re z_1 < \Re z_2 \cdot \ln(t\Re z_2),\ \Re z_2 > 0\Big\}.$$

The group $\mathrm{Aut}(U_{s,t})$ is connected and is generated by one-parameter subgroups of the form

$$\begin{aligned}
\sigma^1_t(z_1, z_2) &= (\exp(t)\ z_1 + t\exp(t)\ z_2, \exp(t)\ z_2),\\
\sigma^2_\beta(z_1, z_2) &= (z_1 + i\beta, z_2),\\
\sigma^3_\gamma(z_1, z_2) &= (z_1, z_2 + i\gamma)
\end{aligned}$$

where $t, \beta, \gamma \in \mathbf{R}$. The holomorphic vector fields corresponding to these are

$$X = (z_1 + z_2)\,\partial/\partial z_1 + z_2\,\partial/\partial z_2,\ \ Y = i\,\partial/\partial z_1,\ \ Z = i\,\partial/\partial z_2$$

respectively and they satisfy the relations

$$[X, Y] = -Y, \quad [Y, Z] = 0, \quad [Z, X] = Y + Z.$$

The commutator $\mathrm{Aut}(U_{s,t})' \backsimeq (\mathbf{R}^2, +)$ is generated by $\sigma^2_\beta, \sigma^3_\gamma$ and it is evident from the relations between $X, Y, Z$ that the subgroup $(\sigma_\beta) \subset \mathrm{Aut}(U_{s,t})'$ is normal in $\mathrm{Aut}(U_{s,t})$. Moreover, the Lie algebra $\mathfrak{g}(U_{s,t})$ has a unique two dimensional abelian subalgebra, namely the one generated by $Y, Z$.

• Fix $b > 0$ and for $0 < t < \infty$ and $\exp(-2\pi b)t < s < t$ define

$$V_{b,s,t} = \Big\{(z_1, z_2) \in \mathbf{C}^2 : s\exp(b\phi) < r < t\exp(b\phi)\Big\}$$

where $(r, \phi)$ are polar coordinates in the $(\Re z_1, \Re z_2)$ plane and $\phi \in (-\infty, \infty)$. The group $\mathrm{Aut}(V_{b,s,t})$ is connected and is generated by one-parameter subgroups of the form

$$\begin{aligned}
&\sigma^1_\psi(z_1, z_2) = (\exp(b\psi)\cos\psi\; z_1 + \exp(b\psi)\sin\psi\; z_2, -\exp(b\psi)\sin\psi\; z_1 + \exp(b\psi)\cos\psi\; z_2)\\
&\sigma^2_\beta(z_1, z_2) = (z_1 + i\beta, z_2)\\
&\sigma^3_\gamma(z_1, z_2) = (z_1, z_2 + i\gamma)
\end{aligned}$$

where $\psi, \beta, \gamma \in \mathbf{R}$. The holomorphic vector fields corresponding to these are

$$X = (bz_1 + z_2)\,\partial/\partial z_1 + (-z_1 + bz_2)\,\partial/\partial z_2, \;\; Y = i\,\partial/\partial z_1, \;\; Z = i\,\partial/\partial z_2$$

respectively and they satisfy the relations

$$[X, Y] = -bY + Z, \quad [Y, Z] = 0, \quad [Z, X] = Y + bZ.$$

As before $\mathrm{Aut}(V_{b,s,t})' \backsimeq (\mathbf{R}^2, +)$ is generated by $\sigma^2_\beta, \sigma^3_\gamma$. The matrix formed by the non-trivial structure constants

$$M = \begin{pmatrix} -b & 1 \\ 1 & b \end{pmatrix}$$

is triangularisable over $\mathbf{R}$ and hence there is a change of coordinates involving $Y, Z$ (and which does not affect $X$) after which it is possible to conclude that there is a one-parameter subgroup in $\mathrm{Aut}(V_{b,s,t})'$ that is normal in $\mathrm{Aut}(V_{b,s,t})$. Finally, the Lie algebra $\mathfrak{g}(V_{b,s,t})$ contains a unique two dimensional abelian subalgebra, namely the one generated by $Y, Z$.

**Proposition 4.5.** *$D$ cannot be equivalent to either $U_{s,t}$ or $V_{b,s,t}$.*

*Proof.* The arguments for either of $U_{s,t}$ or $V_{b,s,t}$ are the same and it will suffice to work with say $V_{b,s,t}$. So let

$$f : V_{b,s,t} \to G \backsimeq D$$

be a biholomorphism where $G$ is as in (2.3). For a subgroup $H$ of $\mathrm{Aut}(V_{b,s,t})$ let $f_*H \subset \mathrm{Aut}(G)$ be the subgroup given by

$$f_*H = \{f \circ h \circ f^{-1} : h \in H\}$$

and likewise define $f^*S = (f^{-1})_*S$ if $S$ is a subgroup of $\mathrm{Aut}(G)$. Let $(\psi_t)$ be the one-parameter subgroup in $\mathrm{Aut}(V_{b,s,t})'$ that is normal in $\mathrm{Aut}(V_{b,s,t})$. Recall that $G$ is invariant under the translations $(T_t)$ along the imaginary $z_2$-axis.

Suppose that $(T_t)$ is not contained in $f_*(\mathrm{Aut}(V_{b,s,t})') \backsimeq (\mathbf{R}^2, +)$. Let $N$ be the subgroup of $\mathrm{Aut}(G)$ that is generated by $f_*(\psi_t)$ and $(T_t)$. Then $N$ is non-abelian and it contains $f_*(\psi_t)$ as a normal subgroup. This situation cannot happen as the proof of proposition 3.2 shows. So $(T_t) \subset f_*(\mathrm{Aut}(V_{b,s,t})') \backsimeq (\mathbf{R}^2, +)$. By proposition 3.1 it follows that

$$G \backsimeq \tilde{G} = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P(\Re z_1) < 0\}$$

after a global change of coordinates and $P(\Re z_1)$ is a polynomial without harmonic terms that depends only on $\Re z_1$, i.e., $\tilde{G}$ is itself a tube domain. Let $f$ still denote the equivalence between $V_{b,s,t}$ and $\tilde{G}$. Evidently $f$ induces an isomorphism of the corresponding Lie algebras. Since there is a unique two dimensional abelian subalgebra in $\mathfrak{g}(V_{b,s,t})$, the same must be true of $\mathfrak{g}(\tilde{G})$. But the subalgebra generated by the translations along the imaginary $z_1, z_2$-axes is one such in $\mathfrak{g}(\tilde{G})$ and so it must be the only one. Thus the corresponding unique two dimensional abelian subalgebras must be mapped into each other. These abelian subalgebras are the Lie algebras of the subgroups formed by translations along the imaginary

$z_1, z_2$-axes in both $V_{b,s,t}$ and $\tilde{G}$. Being isomorphic to $(\mathbf{R}^2, +)$ they are simply connected and hence the isomorphism between the Lie algebras extends to an isomorphism between the subgroups of imaginary translations. It follows (see for example [37] or [34]) that

$$f : V_{b,s,t} \to \tilde{G}$$

must be affine. This however cannot happen as $\partial\tilde{G}$ is algebraic (even polynomial) while $\partial V_{b,s,t}$ is evidently not.

$\square$

4.3. **Another tube domain and its finite and infinite sheeted covers:** For $0 \le s < t < \infty$ define

$$\mathfrak{S}_{s,t} = \Big\{(z_1, z_2) \in \mathbf{C}^2 : s < (\Re z_1)^2 + (\Re z_2)^2 < t\Big\}$$

and for $0 < t < \infty$ define

$$\mathfrak{S}_t = \Big\{(z_1, z_2) \in \mathbf{C}^2 : (\Re z_1)^2 + (\Re z_2)^2 < t\Big\}.$$

Both $\mathrm{Aut}(\mathfrak{S}_{s,t})^c$ and $\mathrm{Aut}(\mathfrak{S}_t)^c$ are generated by maps of the form

$$\begin{aligned}
\sigma^1_\psi(z_1, z_2) &= (\cos\psi\ z_1 - \sin\psi\ z_2, \cos\psi\ z_1 + \sin\psi\ z_2)\\
\sigma^2_\beta(z_1, z_2) &= (z_1 + i\beta, z_2)\\
\sigma^3_\gamma(z_1, z_2) &= (z_1, z_2 + i\gamma)
\end{aligned}$$

where $\psi, \beta, \gamma \in \mathbf{R}$. The holomorphic vector fields corresponding to these are

$$X = -z_2\ \partial/\partial z_1 + z_1\ \partial/\partial z_2,\ \ Y = i\ \partial/\partial z_1,\ \ Z = i\ \partial/\partial z_2$$

respectively and they satisfy the relations

$$[X, Y] = -Z, \quad [Y, Z] = 0, \quad [Z, X] = -Y.$$

Hence the one-parameter subgroup corresponding to $Y + Z$, for example, is contained in the commutator of $\mathrm{Aut}(\mathfrak{S}_t)^c$ and is normal in $\mathrm{Aut}(\mathfrak{S}_t)^c$. As before there is a unique two dimensional abelian subalgebra in $\mathfrak{g}(\mathfrak{S}_t)$, namely that generated by $Y, Z$.

**Proposition 4.6.** *$D$ cannot be equivalent to either $\mathfrak{S}_{s,t}$ or $\mathfrak{S}_t$ for any $0 \le s < t < \infty$.*

*Proof.* The domain $\mathfrak{S}_{s,t}$ is not simply connected and hence $D$ cannot be equivalent to it. On the other hand, the properties of $\mathfrak{S}_t$ as listed above are similar to those of $U_{s,t}, V_{b,s,t}$. The arguments of proposition 4.5 can be applied in this case as well and they show that if $D \backsimeq \mathfrak{S}_t$, then there is an affine equivalence between $\tilde{G}$ and $\mathfrak{S}_t$. However, the explicit description of $\partial\tilde{G}$ and $\partial\mathfrak{S}_t$ shows that this is not possible. $\square$

Next finite and infinite sheeted covers of $\mathfrak{S}_{s,t}$ can be considered. We start with the finite case first. For an integer $n \ge 2$, consider the $n$-sheeted covering self map $\Phi^{(n)}_\chi$ of $\mathbf{C}^2 \setminus \{\Re z_1 = \Re z_2 = 0\}$ whose components are given by

$$\begin{aligned}
\tilde{z}_1 &= \Re\Big((\Re z_1 + i\Re z_2)^n\Big) + i\Im z_1,\\
\tilde{z}_2 &= \Im\Big((\Re z_1 + i\Re z_2)^n\Big) + i\Im z_2.
\end{aligned}$$

Let $M^{(n)}_\chi$ denote the domain of $\Phi^{(n)}_\chi$ endowed with the pull-back complex structure using $\Phi^{(n)}_\chi$. For $0 \le s < t < \infty$ and $n \ge 2$ define

$$\mathfrak{S}^{(n)}_{s,t} = \Big\{(z_1, z_2) \in M^{(n)}_\chi : s^{1/n} < (\Re z_1)^2 + (\Re z_2)^2 < t^{1/n}\Big\}.$$

It can be seen that $\mathfrak{S}^{(n)}_{s,t}$ is an $n$-sheeted cover of $\mathfrak{S}_{s,t}$, the holomorphic covering map being $\Phi^{(n)}_\chi$.

**Proposition 4.7.** *$D$ cannot be equivalent to $\mathfrak{S}^{(n)}_{s,t}$ for $n \ge 2$ and $0 \le s < t < \infty$.*

*Proof.* Let $f : D \to \mathfrak{S}^{(n)}_{s,t}$ be a biholomorphism. Since $\mathfrak{S}^{(n)}_{s,t}$ inherits the pull-back complex structure using $\Phi^{(n)}_\chi$ it follows that

$$\Phi^{(n)}_\chi \circ f : D \to \mathfrak{S}_{s,t}$$

is an unbranched, proper holomorphic mapping between domains that are equipped with the standard complex structure. First suppose that $0 < s < t < \infty$. The boundary $\partial\mathfrak{S}_{s,t}$ has two components, namely

$$\partial\mathfrak{S}^+_{s,t} = \{(z_1, z_2) \in \mathbf{C}^2 : (\Re z_1)^2 + (\Re z_2)^2 = t\}, \text{ and } \partial\mathfrak{S}^-_{s,t} = \{(z_1, z_2) \in \mathbf{C}^2 : (\Re z_1)^2 + (\Re z_2)^2 = s\}.$$

Using the orientation induced on these by $\mathfrak{S}_{s,t}$, it can be seen that $\partial\mathfrak{S}^\pm_{s,t}$ are strongly pseudoconvex and strongly pseudoconcave hypersurfaces respectively. For brevity let $\pi = \Phi^{(n)}_\chi \circ f$ and work near $p_\infty \in \partial D$, the orbit accumulation point. By lemma 2.1 there is a two dimensional stratum $S$ of the Levi degenerate points that clusters at $p_\infty$ and $S \subset \hat{D}$. Choose $a \in S$ near $p_\infty$ and fix a neighbourhood $U$ of $a$ small enough so that $\pi$ extends holomorphically to $U$. Note that $(U \cap \partial D) \setminus S$ consists of either strongly pseudoconvex or strongly pseudoconcave points. Since $\partial D$ is of finite type near $p_\infty$ and $\partial\mathfrak{S}^\pm_{s,t}$ are strongly pseudoconvex/pseudoconcave everywhere, the invariance property of Segre varieties shows that $\pi$ is proper near $a$. Now suppose that $\pi(a) \in \partial\mathfrak{S}^+_{s,t}$. Since $\pi : D \to \mathfrak{S}_{s,t}$ is proper it follows that $\pi(U \cap \partial D) \subset \partial\mathfrak{S}^+_{s,t}$. But then $\pi$ is proper near $a$ as well and hence there is an open dense set of strongly pseudoconcave points in $(U \cap \partial D) \setminus S$ that are mapped locally biholomorphically to points in $\partial\mathfrak{S}^+_{s,t}$ and this contradicts the invariance of the Levi form. Hence $\pi(a) \notin \partial\mathfrak{S}^+_{s,t}$. The same argument shows that $\pi(a) \notin \partial\mathfrak{S}^-_{s,t}$.

The only possibility then is that there are no pseudoconcave points near $p_\infty$, i.e., the boundary $\partial D$ is weakly pseudoconvex near $p_\infty$. In this case it follows from [6] that $D$ is equivalent to the model domain at $p_\infty$ which means that

$$D \backsimeq \tilde{D} = \Big\{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + P_{2m}(z_1, \overline{z}_1) < 0\Big\} \tag{4.2}$$

where $P_{2m}(z_1, \overline{z}_1)$ is a homogeneous subharmonic polynomial of degree $2m$ (this being the 1-type of $\partial D$ at $p_\infty$) without harmonic terms. In particular $D$ is globally pseudoconvex and since $\pi : D \to \mathfrak{S}_{s,t}$ is proper it follows that $\mathfrak{S}_{s,t}$ is also pseudoconvex. This evidently is not the case.

Now suppose that $0 = s < t < \infty$. The two components of the boundary of $\mathfrak{S}_{0,t}$ are

$$\partial\mathfrak{S}^+_{0,t} = \{(z_1, z_2) \in \mathbf{C}^2 : (\Re z_1)^2 + (\Re z_2)^2 = t\}, \text{ and } i\mathbf{R}^2 = \{\Re z_1 = \Re z_2 = 0\}.$$

Choose neighbourhoods $U, U'$ of $a, \pi(a)$ respectively so that $\pi : U \to U'$ is a well defined holomorphic mapping. Suppose that $\pi(a) \in i\mathbf{R}^2$. Let $Z_\pi \subset U$ be the closed analytic set defined by the vanishing of the Jacobian determinant of $\pi$. Note that if it is non-empty, $Z_\pi \subset U \setminus \overline{D}$ since $\pi : D \to \mathfrak{S}_{0,t}$ is unbranched. Since $\partial D$ is of finite type near $p_\infty$ it follows that no open piece of $Z_\pi$ can be contained in $\partial D$. Then there are points in $U \cap \partial D$ that are mapped locally biholomorphically to points in $i\mathbf{R}^2$. This is not possible. For similar reasons $\pi(a) \notin \partial\mathfrak{S}^+_{0,t}$. Hence the only possibility is that $\partial D$ is weakly pseudoconvex near $p_\infty$. As before it follows from [6] that $D \backsimeq \tilde{D}$ where $\tilde{D}$ is as in (4.2). Let $\pi$ still denote the proper mapping

$$\pi : \tilde{D} \to \mathfrak{S}_{0,t}.$$

*Claim:* There exists a point on $\partial\tilde{D}$ whose cluster set under $\pi$ intersects $\partial\mathfrak{S}^+_{0,t}$.

This may be proved as follows by a suitable adaption of the ideas that were used in [10]. Pick $a' \in \partial\mathfrak{S}^+_{0,t}$ and let $\psi$ be a local holomorphic peak function at $a'$. Choose a neighbourhood $U'$ of $a'$ such that $\psi(a') = 1$ and $|\psi| < 1$ on $(\overline{U}' \cap \overline{\mathfrak{S}}_{0,t}) \setminus \{a'\}$. Then $|\psi| \le 1 - 2\alpha$ on $\partial U' \cap \overline{\mathfrak{S}}_{0,t}$ for some $\alpha > 0$. Choose $b' \in U'$ such that $|\psi(b')| = 1 - \alpha$ and let $b \in \tilde{D}$ be such that $\pi(b) = b'$. By proposition 1.1 of [10] there is a smooth analytic disc $h : \overline{\Delta}(0, r) \to \tilde{D}$ such that $h(0) = b$ and $h(\partial\Delta(0, r)) \subset \partial\tilde{D}$. Let $E \subset \Delta(0, r)$ be the connected component of $(\pi \circ h)^{-1}(U' \cap \mathfrak{S}_{0,t})$ that contains the origin. If $E$ is compactly contained in $\Delta(0, r)$ then $\psi \circ \pi \circ h$ is subharmonic on $\overline{E}$ and $|\psi \circ \pi \circ h| \le 1 - 2\alpha$ on $\partial E$ while

$$|\psi \circ \pi \circ h(0)| = |\psi(b')| = 1 - \alpha$$

which contradicts the maximum principle. So there is a sequence $\lambda_j \in E$ that converges to $\lambda_0 \in \partial\Delta(0,r)$. Then $\pi \circ h(\lambda_j) \subset U' \cap \mathfrak{S}_{0,t}$ for all $j$ and if $h(\lambda_j) \to h(\lambda_0) \in \partial\tilde{D}$ it follows that the cluster set of $h(\lambda_0)$ under $\pi$ contains a point in $\overline{U}' \cap \partial\mathfrak{S}_{0,t}$.

It is now known that $\pi$ extends continuously up to $\partial\tilde{D}$ near $h(\lambda_0)$. This extension is even locally biholomorphic across strongly pseudoconvex points that are known to be dense on $\partial\tilde{D}$ by [38]. Now [45] shows that $\pi$ is algebraic. Denote the coordinates in the domain and the range of $\pi$ by $z = (z_1, z_2)$ and $w = (w_1, w_2)$ respectively. Then there are polynomials

$$P_1(z_1, w_1, w_2) = a_k(w_1, w_2)z_1^k + a_{k-1}(w_1, w_2)z_1^{k-1} + \ldots + a_0(w_1, w_2),$$
$$P_2(z_2, w_1, w_2) = b_l(w_1, w_2)z_2^l + b_{l-1}(w_1, w_2)z_2^{l-1} + \ldots + b_0(w_1, w_2)$$

of degree $k, l$ respectively and the coefficients $a_\mu, b_\nu$ are polynomials in $w_1, w_2$, with the property that $P_1(z_1, w_1, w_2) = P_2(z_2, w_1, w_2) = 0$ whenever $\pi^{-1}(w) = z$. Away from an algebraic variety $V_1 \subset \mathbf{C}^2_w$ of dimension one, the zero locus of these polynomials defines a correspondence $\hat{F} : \mathbf{C}^2_w \to \mathbf{C}^2_z$ such that $\mathrm{Graph}(\pi^{-1}) \subset \mathfrak{S}_{0,t} \times \tilde{D}$ is an irreducible component of $\mathrm{Graph}(\hat{F}) \cap (\mathfrak{S}_{0,t} \times \tilde{D})$. Let $V_2 \subset \mathbf{C}^2_w$ be the branching locus of $\hat{F}$ which is again of dimension one. Note that near each point of $\mathbf{C}^2_w \setminus (V_1 \cup V_2)$ the correspondence $\hat{F}$ splits as the union of locally well defined holomorphic functions. Also observe that the real dimension of $i\mathbf{R}^2 \cap (V_1 \cup V_2)$ is at most one. So pick $a' \in i\mathbf{R}^2 \setminus (V_1 \cup V_2)$ and choose a neighbourhood $U'$ of $a'$ that does not intersect $V_1 \cup V_2$. All branches of $\hat{F}$ are then well defined on $U'$ and some of these are exactly the various branches of $\pi^{-1}$. Let $\pi_1^{-1}$ be one such branch of $\pi^{-1}$ that is holomorphic on $U'$. Since $\pi$ is globally proper, $\pi_1^{-1}$ maps $U' \cap i\mathbf{R}^2$ into $\partial\tilde{D}$. The branch locus of $\pi_1^{-1}$, if non-empty has dimension at most one since $\pi$ is unbranched. Therefore the intersection of the branch locus with $i\mathbf{R}^2$ has real dimension at most one. Hence it is possible to find points on $i\mathbf{R}^2$ near $a'$ that are mapped locally biholomorphically by $\pi_1^{-1}$ to points on $\partial\tilde{D}$. This cannot hold as $\partial\tilde{D}$ is not totally real. $\square$

Next consider the infinite sheeted covering

$$\Phi_\chi^{(\infty)} : \mathbf{C}^2 \to \mathbf{C}^2 \setminus \{\Re z_1 = \Re z_2 = 0\}$$

whose components are given by

$$\tilde{z}_1 = \exp(\Re z_1)\cos(\Im z_1) + i\Re z_2,$$
$$\tilde{z}_2 = \exp(\Re z_1)\sin(\Im z_1) + i\Im z_2.$$

The domain of $\Phi_\chi^{(\infty)}$ is equipped with the pull-back complex structure and the resulting complex manifold is denoted by $M_\chi^{(\infty)}$. For $0 \le s < t < \infty$ define

$$\mathfrak{S}_{s,t}^{(\infty)} = \Big\{(z_1, z_2) \in M_\chi^{(\infty)} : (\ln s)/2 < \Re z_1 < (\ln t)/2\Big\}$$

and this is seen to be an infinite sheeted covering of $\mathfrak{S}_{s,t}$, the holomorphic covering map being $\Phi_\chi^{(\infty)}$.

**Proposition 4.8.** *$D$ cannot be equivalent to $\mathfrak{S}_{s,t}^{(\infty)}$ for $0 \le s < t < \infty$.*

*Proof.* Let $f : D \to \mathfrak{S}_{s,t}^{(\infty)}$ be a biholomorphism. As before

$$\Phi_\chi^{(\infty)} \circ f : D \to \mathfrak{S}_{s,t}$$

is then a holomorphic infinite sheeted covering (in particular it is non-proper) between domains with the standard complex structure. The explicit description of $\Phi_\chi^{(\infty)}$ shows that $\Phi_\chi^{(\infty)}\Big(\partial\mathfrak{S}_{s,t}^{(\infty)}\Big) \subset \partial\mathfrak{S}_{s,t}$ and this implies that the cluster set of $\partial D$ under $\pi = \Phi_\chi^{(\infty)} \circ f$ is contained in $\partial\mathfrak{S}_{s,t}$. This observation is an effective replacement for the properness of $\pi$ in the previous proposition.

In case $0 < s < t < \infty$, this observation allows the use of the arguments in the previous proposition and they show that

$$\pi : D \backsimeq \tilde{D} \to \mathfrak{S}_{s,t}$$

is a covering with $\tilde{D}$ as in (4.2). Note that the Kobayashi metric on $\tilde{D}$ is complete and hence the same holds for the base $\mathfrak{S}_{s,t}$. Completeness then forces $\mathfrak{S}_{s,t}$ to be pseudoconvex which however is not the

case. Now suppose that $0 = s < t < \infty$. Again the arguments used before apply, including the claim made there. So there are points on $\partial\tilde{D}$ whose cluster set under the covering map $\pi$ intersects $\partial\mathfrak{S}^+_{0,t}$. It follows from [42] that $\pi$ extends continuously up to $\partial\tilde{D}$ near such points. Moreover the extension is locally proper near $\partial\tilde{D}$. The rest of the proof proceeds exactly as in the previous proposition. $\square$

4.4. **A domain in $\mathbf{P}^2$:** Let $\mathcal{Q}_+ \subset \mathbf{C}^3$ be the smooth complex analytic set given by

$$z_0^2 + z_1^2 + z_2^2 = 1$$

and for $t \geq 0$ consider the spheres

$$\Sigma_t = \Big\{(z_0, z_1, z_2) \in \mathbf{C}^3 : |z_0|^2 + |z_1|^2 + |z_2|^2 = t\Big\}.$$

Note that $\Sigma_t \cap \mathcal{Q}_+ = \emptyset$ for $0 \leq t < 1$. If $t > 1$ it can be checked that $\Sigma_t$ intersects $\mathcal{Q}_+$ transversally everywhere. On the other hand $\Sigma_1 \cap \mathcal{Q}_+ = \mathbf{R}^3 \cap \mathcal{Q}_+$ which is totally real in $\mathbf{C}^3$.

For $1 \leq s < t < \infty$ define

$$E^{(2)}_{s,t} = \Big\{(z_0, z_1, z_2) \in \mathbf{C}^3 : s < |z_0|^2 + |z_1|^2 + |z_2|^2 < t\Big\} \cap \mathcal{Q}_+.$$

This domain is a 2-sheeted covering of

$$E_{s,t} = \Big\{[z_0 : z_1 : z_2] \in \mathbf{P}^2 : s\,|z_0^2 + z_1^2 + z_2^2| < |z_0|^2 + |z_1|^2 + |z_2|^2 < t\,|z_0^2 + z_1^2 + z_2^2|\Big\}$$

via the natural map $\psi(z_0, z_1, z_2) = [z_0 : z_1 : z_2]$. In the same way, for $1 < t < \infty$ the domain

$$E_t = \Big\{[z_0 : z_1 : z_2] \in \mathbf{P}^2 : |z_0|^2 + |z_1|^2 + |z_2|^2 < t\,|z_0^2 + z_1^2 + z_2^2|\Big\}$$

is covered in a $2 : 1$ manner by

$$E^{(2)}_t = \Big\{(z_0, z_1, z_2) \in \mathbf{C}^3 : |z_0|^2 + |z_1|^2 + |z_2|^2 < t\Big\} \cap \mathcal{Q}_+$$

by the same map $\psi$. Note that $E_{1,t} \cup \psi(\Sigma_1 \cap \mathcal{Q}_+) = E_t$ for all $1 < t < \infty$.

Consider the map $\Phi_\mu : \mathbf{C}^2 \setminus \{0\} \to \mathcal{Q}_+$ given by

$$\begin{aligned}
\tilde{z}_1 &= -i(z_1^2 + z_2^2) + i(z_1\overline{z}_2 - \overline{z}_1 z_2)/(|z_1|^2 + |z_2|^2)\\
\tilde{z}_2 &= z_1^2 - z_2^2 - (z_1\overline{z}_2 + \overline{z}_1 z_2)/(|z_1|^2 + |z_2|^2)\\
\tilde{z}_3 &= 2z_1z_2 + (|z_1|^2 - |z_2|^2)/(|z_1|^2 + |z_2|^2)
\end{aligned}$$

which is a two sheeted covering onto $\mathcal{Q}_+ \setminus \mathbf{R}^3$. The domain of $\Phi_\mu$ is now equipped with the pull back complex structure using $\Phi_\mu$ and the resulting complex manifold is denoted by $M^{(4)}_\mu$. For $1 \leq s < t < \infty$ define

$$E^{(4)}_{s,t} = \Big\{(z_1, z_2) \in M^{(4)}_\mu : \sqrt{(s-1)/2} < |z_1|^2 + |z_2|^2 < \sqrt{(t-1)/2}\Big\}.$$

Note that $E^{(4)}_{s,t}$ is a four-sheeted cover of $E_{s,t}$, the covering map being $\psi \circ \Phi_\mu$.

**Proposition 4.9.** *There cannot exist a proper holomorphic mapping between $D$ and $E_{s,t}$ for $1 \leq s < t < \infty$. In particular $D$ cannot be equivalent to either $E^{(2)}_{s,t}$ or $E^{(4)}_{s,t}$ for $1 \leq s < t < \infty$.*

*Proof.* Let $f : D \to E_{s,t}$ be a proper holomorphic mapping. First consider the case when $1 < s < t < \infty$. The boundary of $E_{s,t}$ has two components which are covered in a 2:1 manner by $\Sigma_t \cap \mathcal{Q}_+$ and $\Sigma_s \cap \mathcal{Q}_+$ respectively using $\psi$. Let

$$\begin{aligned}
\partial E^+_{s,t} &= \Big\{[z_0 : z_1 : z_2] \in \mathbf{P}^2 : |z_0|^2 + |z_1|^2 + |z_2|^2 = t|z_0^2 + z_1^2 + z_2^2|\Big\} = \psi(\Sigma_t \cap \mathcal{Q}_+), \text{ and}\\
\partial E^-_{s,t} &= \Big\{[z_0 : z_1 : z_2] \in \mathbf{P}^2 : |z_0|^2 + |z_1|^2 + |z_2|^2 = t|z_0^2 + z_1^2 + z_2^2|\Big\} = \psi(\Sigma_s \cap \mathcal{Q}_+)
\end{aligned}$$

which are strongly pseudoconvex and strongly pseudoconcave hypersurfaces respectively. Once again the structure of $\partial D$ near $p_\infty$ can be exploited exactly as in proposition 4.7, the conclusion being that $\partial D$ must be weakly pseudoconvex near $p_\infty$. By [6] it follows that $D \backsimeq \tilde{D}$ where $\tilde{D}$ is as in (4.2). Thus

$$f : D \backsimeq \tilde{D} \to E_{s,t}$$

is proper. Now all branches of $f^{-1}$ will extend across $\partial E^-_{s,t}$ and the extension will map $\partial E^-_{s,t}$ into $\partial\tilde{D}$. This implies that there are strongly pseudoconcave points on $\partial\tilde{D}$ which is not possible.

Now suppose that $1 = s < t < \infty$. Then

$$\partial E_{1,t} = \partial E^+_{1,t} \cup \psi(\Sigma_1 \cap \mathcal{Q}_+)$$

where $\psi(\Sigma_1 \cap \mathcal{Q}_+)$ is maximally totally real. Let $\phi$ be a holomorphic function on $\tilde{D}$ that peaks at the point at infinity in $\partial\tilde{D}$ (see [1] for example) and denote by $f_1^{-1}, f_2^{-1}, \ldots, f_l^{-1}$ the locally defined branches of $f^{-1}$ that exist away from a closed analytic set of dimension one in $E_{1,t}$. Then

$$\tilde{\phi} = (\phi \circ f_1^{-1}) \cdot (\phi \circ f_2^{-1}) \cdot \ldots \cdot (\phi \circ f_l^{-1})$$

is a well defined holomorphic function on $E_{1,t}$ such that $|\tilde{\phi}| < 1$ there.

*Claim:* For each $p' \in \psi(\Sigma_1 \cap \mathcal{Q}_+)$ there exists $p \in \partial\tilde{D}$ such that the cluster set of $p$ under $f$ contains $p'$.

Indeed it has been noted that $E_{1,t} \cup \psi(\Sigma_1 \cap \mathcal{Q}_+) = E_t$ and that $\psi(\Sigma_1 \cap \mathcal{Q}_+)$ has real codimension two. Therefore $\tilde{\phi} \in \mathcal{O}(E_{1,t})$ extends holomorphically to $E_t$, and the extension that is still denoted by $\tilde{\phi}$ satisfies $|\tilde{\phi}| \le 1$ there. If $|\tilde{\phi}(p')| = 1$ for some $p' \in \psi(\Sigma_1 \cap \mathcal{Q}_+)$ then the maximum principle implies that $|\tilde{\phi}| \equiv 1$ on $E_{1,t} \subset E_t$ and this is a contradiction. The claim follows.

The claim made in proposition 4.7 holds here as well; indeed the only property in the range space that is used is the existence of a local holomorphic peak function and these exist near each point of $\partial E^+_{1,t}$. Therefore there are points $a \in \partial\tilde{D}$ and $a' \in \partial E^+_{1,t}$ such that the cluster set of $a$ under $f$ contains $a'$. Working in a coordinate system around $a'$ it is known that (see for example [42] or [11]) for sufficiently small neighbourhoods $U, U'$ around $a, a'$ respectively,

$$\mathrm{dist}(f(z), U' \cap \partial E^+_{1,t}) \lesssim \mathrm{dist}(z, U \cap \partial\tilde{D}) \tag{4.3}$$

whenever $z \in U \cap \tilde{D}$ is such that $f(z) \in U' \cap E^+_{1,t}$. Using the estimates on the infinitesimal Kobayashi metric near $a, a'$ it follows that $f$ extends continuously up to $\partial\tilde{D}$ near $a$ and $f(a) = a'$. Therefore $f$ extends locally biholomorphically across all the strongly pseudoconvex points near $a$. Let $w = (w_1, w_2)$ denote affine coordinates near $a'$ and $f_1^{-1}$ be a branch of $f^{-1}$ that extends locally biholomorphically across near $a'$. Then by [45] the graph of $f_1^{-1}$ near $(a, a')$ is contained in the zero locus of the polynomials $P_1(z_1, w_1, w_2), P_2(z_2, w_1, w_2)$ as in proposition 4.7 and where $f_1^{-1}(w) = (z_1, z_2) = z$. Projectivize these polynomials by replacing $w_i \mapsto w_i/w_0, z_i \mapsto z_i/z_0$ for $i = 1, 2$ to get an algebraic variety $A \subset \mathbf{P}^2_z \times \mathbf{P}^2_w$ such that Graph$(f)$ is an irreducible component of $A \cap (\tilde{D} \times E_{1,t})$. Away from an algebraic variety $V \subset \mathbf{P}^2_w$ of dimension one, $A$ is the union of the graphs of locally well defined holomorphic functions. Since $\psi(\Sigma_1 \cap \mathcal{Q}_+)$ is totally real it follows that $\psi(\Sigma_1 \cap \mathcal{Q}_+) \cap V$ has real dimension at most one. Choose $p' \in \psi(\Sigma_1 \cap \mathcal{Q}_+) \setminus V$ and a neighbourhood $U'$ of $p'$ that does not intersect $V$. Then all branches of the correspondence defined by $A$ are well defined in $U'$ and some of these will coincide with those of $f^{-1}$. By the claim above, there is at least one branch of $f^{-1}$, call it $\tilde{f}^{-1}$ such that $\tilde{f}^{-1}(p') \notin H_0 = \{z_0 = 0\}$ the hyperplane at infinity in $\mathbf{P}^2_z$, i.e., $\tilde{f}^{-1}(p') \in \partial\tilde{D}$. Therefore near $p'$ there are points on $\psi(\Sigma_1 \cap \mathcal{Q}_+)$ that are mapped locally biholomorphically by $\tilde{f}^{-1}$ to points on $\partial\tilde{D}$ and this is a contradiction since $\partial\tilde{D}$ is not totally real.

To conclude, if $D \backsimeq E^{(2)}_{s,t}$ or $E^{(4)}_{s,t}$ then this would imply the existence of an unbranched proper mapping between $D$ and $E_{s,t}$ and this cannot happen by the arguments given above. $\square$

**Proposition 4.10.** *There cannot exist a proper holomorphic mapping between $D$ and $E_t$ for $1 < t < \infty$. In particular $D$ cannot be equivalent to $E^{(2)}_t$ for $1 < t < \infty$.*

*Proof.* Let $f : D \to E_t$ be proper. Working in affine coordinates near each point in $\partial E_t$ it can be seen that $E_t$ is described by $\{\varrho(z) < 0\}$ where $\varrho(z) = 1 + |z_1|^2 + |z_2|^2 - t|1 + z_1^2 + z_2^2|$ is strongly plurisubharmonic. It follows that $E_t$ must be holomorphically convex (see for example [19]) and thus $D$ is pseudoconvex. Choose $p_j \in D$ converging to $p_\infty$ and let $f(p_j) \to p'_\infty \in \partial E_t$. As in the previous proposition, by working in affine coordinates around $p'_\infty$ it is possible to show that (4.3) holds near $p_\infty, p'_\infty$. Hence $f$ extends

continuously up to $\partial D$ near $p_\infty$ with $f(p_\infty) = p'_\infty$. It follows from theorem 1.2 of [11] that $\partial D$ is weakly spherical near $p_\infty$, i.e., the defining function for $\partial D$ near $p_\infty = 0$ has the form

$$\rho(z) = 2\Re z_2 + |z_1|^{2m} + \text{ higher order terms.}$$

Since $p_\infty \in \partial D$ is an orbit accumulation point it follows from [6] that $D$ is equivalent to the model domain at $p_\infty$, i.e., $D \backsimeq \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0\}$. This shows that $\dim \mathrm{Aut}(D) = 4$ which is a contradiction.

To conclude, if $D \backsimeq E_t^{(2)}$ then there would exist an unbranched proper mapping between $D$ and $E_t$ and this is not possible. □

4.5. **Domains constructed by using an analogue of Rossi's map:** Let $\mathcal{Q}_- \subset \mathbf{C}^3$ be the smooth complex analytic set given by

$$z_1^2 + z_2^2 - z_3^2 = 1.$$

Let $\Omega = \Big\{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 - |z_2|^2 \neq 0\Big\}$ and consider the map $\Phi : \Omega \to \mathcal{Q}_-$ given by

$$\begin{aligned}
\tilde{z}_1 &= -i(z_1^2 + z_2^2) - i(z_1\overline{z}_2 + \overline{z}_1 z_2)/(|z_1|^2 - |z_2|^2),\\
\tilde{z}_2 &= z_1^2 - z_2^2 + (z_1\overline{z}_2 - \overline{z}_1 z_2)/(|z_1|^2 - |z_2|^2),\\
\tilde{z}_3 &= -2iz_1z_2 - i(|z_1|^2 + |z_2|^2)/(|z_1|^2 - |z_2|^2).
\end{aligned}$$

It can be checked that $\Phi(\Omega) = \mathcal{Q}_- \setminus (\mathcal{Q}_- \cap \mathcal{W})$ where $\mathcal{W} = i\mathbf{R}^3 \cup \mathbf{R}^3 \cup \Big\{(z_1, z_2, z_3) \in \mathbf{C}^3 \setminus \mathbf{R}^3 : |iz_1 + z_2| = |iz_3 - 1|,\ |iz_1 - z_2| = |iz_3 + 1|\Big\}$. Set

$$\Omega^> = \Big\{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 - |z_2|^2 > 0\Big\}$$

and let $\Phi^>$ be the restriction of $\Phi$ to $\Omega^>$. Then $\Phi^> : \Omega^> \to \Phi^>(\Omega^>)$ is a two sheeted covering where $\Phi^>(\Omega^>)$ is the union of

$$\begin{aligned}
\Sigma^\nu &= \Big\{(z_1, z_2, z_3) \in \mathbf{C}^3 : -1 < |z_1|^2 + |z_2|^2 - |z_3|^2 < 1,\ \Im z_3 < 0\Big\} \cap \mathcal{Q}_-,\\
\Sigma^\eta &= \Big\{(z_1, z_2, z_3) \in \mathbf{C}^3 : |z_1|^2 + |z_2|^2 - |z_3|^2 > 1,\ \Im(z_2(\overline{z}_1 + \overline{z}_3)) > 0\Big\} \cap \mathcal{Q}_-,\\
O_5 &= \Big\{(z_1, z_2, z_3) \in \mathbf{C}^3 \setminus \mathbf{R}^3 : |iz_1 + z_2| = |iz_3 + 1|, |iz_1 - z_2| = |iz_3 - 1|, \Im z_3 < 0\Big\} \cap \mathcal{Q}_-.
\end{aligned}$$

In particular if $\Phi^\nu, \Phi^\eta$ are the restrictions of $\Phi^>$ to $\Omega^\nu = \Big\{(z_1, z_2) \in \mathbf{C}^2 : 0 < |z_1|^2 - |z_1|^2 < 1\Big\}$ and $\Omega^\eta = \Big\{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 - |z_2|^2 > 1\Big\}$ respectively, then $\Phi^\nu : \Omega^\nu \to \Sigma^\nu$ and $\Phi^\eta : \Omega^\eta \to \Sigma^\eta$ are also two sheeted coverings. It follows from the definition that $z_3 \neq 0$ on $\Sigma^\nu$ and therefore $\Psi^\nu : \Sigma^\nu \to \mathbf{C}^2$ given by $\Psi^\nu(z_1, z_2, z_3) = (z_1/z_3, z_2/z_3)$ is holomorphic. Moreover $\Psi^\nu$ is injective as well on $\Sigma^\nu$ for if

$$\Psi^\nu(z_1^*, z_2^*, z_3^*) = (z_1^*/z_3^*, z_2^*/z_3^*) = (z_1/z_3, z_2/z_3) = \Psi^\nu(z_1, z_2, z_3)$$

then $z_1^* = \lambda z_3^*, z_1 = \lambda z_3$ and $z_2^* = \mu z_3^*, z_2 = \mu z_3$ for scalars $\lambda, \mu$. But then $z_1^2 + z_2^2 - z_3^2 = 1 = (z_1^*)^2 + (z_2^*)^2 - (z_3^*)^2$ implies that $z_3 = \pm z_3^*$ and only the first alternative can hold since $\Im z_3$ and $\Im z_3^*$ are both positive. Therefore

$$D^\nu = \Psi^\nu(\Sigma^\nu) = \Big\{(z_1, z_2) \in \mathbf{C}^2 : -|z_1^2 + z_2^2 - 1| < |z_1|^2 + |z_2|^2 - 1 < |z_1^2 + z_2^2 - 1|\Big\} \backsimeq \Sigma^\nu.$$

Likewise note that for $-1 \leq s < t \leq 1$, the domain

$$\Sigma^\nu_{s,t} = \Big\{(z_1, z_2, z_3) \in \mathbf{C}^3 : s < |z_1|^2 + |z_2|^2 - |z_3|^2 < t,\ \Im z_3 < 0\Big\} \cap \mathcal{Q}_- \subset \Sigma^\nu$$

is mapped biholomorphically by $\Psi^\nu$ onto

$$\Omega_{s,t} = \Psi^\nu(\Sigma^\nu_{s,t}) = \Big\{(z_1, z_2) \in \mathbf{C}^2 : s|z_1^2 + z_2^2 - 1| < |z_1|^2 + |z_2|^2 - 1 < t|z_1^2 + z_2^2 - 1|\Big\}.$$

It is also possible to consider the domain

$$\Omega_t = \Big\{(z_1, z_2) \in C^2 : |z_1|^2 + |z_2|^2 - 1 < t|z_1^2 + z_2^2 - 1|\Big\}$$

for $t \in (-1,1)$. It has been observed in [23] that this domain has a unique maximally totally real $\mathrm{Aut}(\Omega_t)^c$-orbit, namely

$$\mathcal{O}_5 = \Big\{ (\Re z_1, \Re z_2) \in \mathbf{R}^2 : (\Re z_1)^2 + (\Re z_2)^2 < 1 \Big\}$$

for all $t \in (-1,1)$. Furthermore note that $\Omega_t = \Omega_{-1,t} \cup \mathcal{O}_5$ for all $t \in (-1,1)$.

On the other hand observe that $z_1 \neq 0$ on $\Sigma^\eta$ and therefore $\Psi^\eta : \Sigma^\eta \to \mathbf{C}^2$ given by $\Psi^\eta(z_1, z_2, z_3) = (z_2/z_1, z_3/z_1)$ is holomorphic. A similar calculation shows that $\Psi^\eta(z_1, z_2, z_3) = \Psi^\eta(-z_1, -z_2, -z_3)$ and therefore

$$\Psi^\eta : \Sigma^\eta \to D^\eta = \Psi^\eta(\Sigma^\eta) = \Big\{ (z_1, z_2) \in \mathbf{C}^2 : |1 + z_1^2 - z_2^2| < 1 + |z_1|^2 - |z_2|^2,\ \Im(z_1(1 + \overline{z}_2)) > 0 \Big\}$$

is a two sheeted covering. Likewise note that for $1 \leq s < t \leq \infty$ the domain

$$\Sigma^\eta_{s,t} = \Big\{ (z_1, z_2, z_3) \in \mathbf{C}^3 : s < |z_1|^2 + |z_2|^2 - |z_3|^2 < t,\ \Im(z_2(\overline{z}_1 + \overline{z}_3)) > 0 \Big\} \cap \mathcal{Q}_- \subset \Sigma^\eta$$

is a two sheeted covering of

$$D_{s,t} = \Psi^\eta(\Sigma^\eta_{s,t}) = \Big\{ (z_1, z_2) \in \mathbf{C}^2 : s|1 + z_1^2 - z_2^2| < 1 + |z_1|^2 - |z_2|^2 < t|1 + z_1^2 - z_2^2|,\ \Im(z_1(1 + \overline{z}_2)) > 0 \Big\}.$$

When $t = \infty$ the domain $D_{s,\infty}$ is assumed not to include the complex curve $\mathcal{O} = \Big\{ (z_1, z_2) \in \mathbf{C}^2 : 1 + z_1^2 - z_2^2 = 0,\ \Im(z_1(1 + \overline{z}_2)) > 0 \Big\}$. It is also possible to consider the domain

$$D_s = \Big\{ (z_1, z_2) \in \mathbf{C}^2 : s|1 + z_1^2 - z_2^2| < 1 + |z_1|^2 - |z_2|^2,\ \Im(z_1(1 + \overline{z}_2)) > 0 \Big\}$$

for $1 \leq s < \infty$. Here $\mathcal{O}$ is allowed to be in $D_s$ and hence $D_s = D_{s,\infty} \cup \mathcal{O}$.

**Lemma 4.11.** *$D$ cannot be equivalent to either $\Omega_{s,t}$ or $D_{s,t}$ for all permissible values of $s,t$.*

*Proof.* First suppose that $D \backsimeq \Omega_{s,t}$. For $-1 \leq s < t \leq 1$ the discussion above shows that

$$\Psi^\nu \circ \Phi^\nu : \Big\{ (z_1, z_2) \in \mathbf{C}^2 : \sqrt{(s+1)/2} < |z_1|^2 - |z_2|^2 < \sqrt{(t+1)/2} \Big\} \to \Omega_{s,t} \backsimeq D$$

is a two sheeted cover. Since $D$ is simply connected it follows that $\Psi^\nu \circ \Phi^\nu$ is a homeomorphism which is evidently not the case.

On the other hand if $1 \leq s < t \leq \infty$, it again follows that

$$\Psi^\eta \circ \Phi^\eta : \Big\{ (z_1, z_2) \in \mathbf{C}^2 : \sqrt{(s+1)/2} < |z_1|^2 - |z_2|^2 < \sqrt{(t+1)/2} \Big\} \to D_{s,t} \backsimeq D$$

is a four sheeted cover. As above, since $D$ is simply connected it follows that $\Psi^\eta \circ \Phi^\eta$ is forced to be a homeomorphism which is not the case. $\square$

**Proposition 4.12.** *There cannot exist a proper holomorpic map from $D$ onto $\Omega_t$ for $t \in (-1,1)$ or to $D_s$ for $1 \leq s < \infty$.*

*Proof.* First work with $\Omega_t$. Write $z_1 = x + iy,\ z_2 = u + iv$ so that $\mathcal{O}_5 = \{(x,u) \in \mathbf{R}^2 : x^2 + u^2 < 1\}$. Note that $\partial\mathcal{O}_5 = \{(x,u) \in \mathbf{R}^2 : x^2 + u^2 = 1\} \subset \partial\Omega_t$ for all $t \in (-1,1)$ and that $\partial\Omega_t \setminus \partial\mathcal{O}_5$ is a smooth strongly pseudoconvex hypersurface. Let $f : D \to \Omega_t$ be proper. As in proposition 4.7, we work near $p_\infty \in \partial D$ and choose $a \in S$, where $S$ is a two dimensional stratum of the Levi degenerate points that clusters at $p_\infty$ and which is contained in $\hat{D}$. Then $f$ extends holomorphically near $a$ and $f(a) \in \partial\Omega_t$. Choose neighbourhoods $U, U'$ of $a, f(a)$ respectively so that $f : U \to U'$ is a well defined holomorphic mapping with $f(U \cap \partial D) \subset U' \cap \partial\Omega_t$. The closed complex analytic set $Z_f \subset U$ defined by the vanishing of the Jacobian of $f$ has dimension one and the finite type assumption on $\partial D$ near $p_\infty$ implies that the real dimension of $Z_f \cap \partial D$ is at most one. Therefore it is possible to choose $p \in (U \cap S) \setminus Z_f$. Suppose that $f(p) \in \partial\Omega_t \setminus \partial\mathcal{O}_5$. Then there are strongly pseudoconcave points near $p$ that are mapped locally biholomorphically to points near $f(p)$ which however is a strongly pseudoconvex point. This cannot happen. The other possibility is that $f(p) \in \partial\mathcal{O}_5 = \{(x,u) \in \mathbf{R}^2 : x^2 + u^2 = 1\}$ which is totally real. It follows that $f^{-1}(U' \cap \partial\mathcal{O}_5) \subset U \cap \partial D$ is nowhere dense and therefore there are strongly pseudoconcave points near $p$ that are mapped locally biholomorphically to strongly pseudoconvex points near $f(p)$. This is again a contradiction.

Hence the boundary $\partial D$ near $p_\infty$ is weakly pseudoconvex and by [6] it follows that $D \backsimeq \tilde{D}$ where $\tilde{D}$ is as in (4.2). Then $f : D \backsimeq \tilde{D} \to \Omega_t$ is still biholomorphic. Observe that $\tilde{D}$ is invariant under the one-parameter subgroups $T_t(z_1, z_2) = (z_1, z_2 + it)$ and $S_s(z_1, z_2) = (\exp(s/2m)z_1, \exp(s)z_2)$ and the corresponding real vector fields are $X = \Re(i\ \partial/\partial z_2)$ and $Y = \Re((z_1/2m)\ \partial/\partial z_1 + z_2 \partial/\partial z_2)$. It can be seen that $[X, Y] = X$. By the reasoning given in the last part of proposition 4.1 it follows that $\tilde{D} \backsimeq \mathcal{D}_4 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + (\Re z_1)^{2m} < 0\}$. Let $f : \mathcal{D}_2 \to \Omega_t$ still denote the proper map. Choose $\tilde{p}' \in \partial\Omega_t \setminus \partial\mathcal{O}_5$ a strongly pseudoconvex point. Then the claim made in proposition 4.7 shows the existence of $\tilde{p} \in \partial\mathcal{D}_2$ such that the cluster set of $\tilde{p}$ under $f$ contains $\tilde{p}'$. Then $f$ will extend continuously up to the boundary $\partial\mathcal{D}_2$ near $\tilde{p}$ and $f(\tilde{p}) = \tilde{p}'$. Now it follows from [11] that $\tilde{p} \in \partial\mathcal{D}_2$ must be a weakly spherical point, i.e., there exists a coordinate system around $\tilde{p}$ such that the defining equation for $\partial\Omega_2$ near $\tilde{p}$ is of the form

$$\rho(z) = 2\Re z_2 + |z_1|^{2m} + \text{ higher order terms.}$$

However the explicit form of $\partial\mathcal{D}_2$ shows that no point on it is weakly spherical.

On the other hand observe that if $s > 1$ then $\partial D_s$ is the disjoint union of

$$\begin{aligned}
\mathcal{C}^1 &= \{1 + |z_1|^2 - |z_2|^2 = s|1 + z_1^2 - z_2^2|,\ \Im(z_1(1 + \overline{z}_2)) > 0\},\\
\mathcal{C}^2 &= \{1 + |z_1|^2 - |z_2|^2 > s|1 + z_1^2 - z_2^2|,\ \Im(z_1(1 + \overline{z}_2)) = 0\},\\
\mathcal{C}^3 &= \{1 + |z_1|^2 - |z_2|^2 = s|1 + z_1^2 - z_2^2|,\ \Im(z_1(1 + \overline{z}_2)) = 0\}.
\end{aligned}$$

For an arbitrary set $E \subset \mathbf{C}^2$ and $e \in E$, let $E_e$ denote the germ of $E$ at $e$. Note that $\mathcal{C}^1$ is a strongly pseudoconvex hypersurface. Also $\Im(z_1(1 + \overline{z}_2)) = 0$ has an isolated singularity at $(z_1, z_2) = (0, -1)$ away from which it is a smooth Levi flat hypersurface. Observe that $(0, -1) \notin \mathcal{C}^2$ since $s > 1$. As above choose neighbourhoods $U, U'$ of $a, f(a)$ respectively and note that the real dimension of $Z_f \cap U$ is at most one. Pick $p \in (U \cap S) \setminus Z_f$. Since $\mathcal{C}^1$ is strongly pseudoconvex it follows that $f(p) \notin \mathcal{C}^1$. Moreover $f(p) \notin \mathcal{C}^2$ since it is Levi flat and all points of $U \setminus S$ are Levi non-degenerate. The only possibility is that $f(p) \in \mathcal{C}^3$. We now work near $p$. Suppose that $f((\partial D)_p) \subset \mathcal{C}^3_{f(p)} \subset \{\Im(z_1(1+\overline{z}_2)) = 0\}_{f(p)}$. The first and third sets are real analytic germs of dimension 3 and hence they must coincide. It follows that $f$ maps a neighbourhood of $p$ locally biholomorphically onto a neighbourhood of $f(p)$ on the Levi flat hypersurface $\{\Im(z_1(1 + \overline{z}_2)) = 0\}$. This cannot happen as there are Levi non-degenerate points near $p$. This means that there is an open dense set of points near $p$ that are mapped by $f$ locally biholomorphically into $\mathcal{C}^1$ or $\mathcal{C}^2$. Both possibilities however violate the invariance of the Levi form. The rest of the argument runs along in the same way as mentioned above.

When $s = 1$ it was observed in [23] that $\Psi_\eta : D_1^{(2)} \to D_1$ is proper holomorphic and that $D_1^{(2)} \backsimeq \Delta^2$ the bidisc. The domain $D_1^{(2)}$ will be discussed later but assuming this for the moment, it follows that if $D \backsimeq D_1$, then there will exist a proper holomorphic mapping from $\Delta^2$ onto $D$. Since there are strongly pseudoconvex points on $\partial D$ near $p_\infty$ it follows from [22] that such a proper mapping cannot exist. □

4.6. **Finite and infinite sheeted covers of $D_{s,t}$, $\Omega_{s,t}$:** Let $[z_0 : z_1 : z_2 : z_3]$ and $[\zeta : z : w]$ denote coordinates on $\mathbf{P}^3$ and $\mathbf{P}^2$ respectively where $\{z_0 = 0\}$ and $\{\zeta = 0\}$ are to be regarded as the respective hyperplanes at infinity. Let $\mathcal{Q}_- \subset \mathbf{P}^3$ be the smooth projective variety given by $z_0^2 = z_1^2 + z_2^2 - z_3^2$ and set $\Sigma = \{[\zeta : z : w] \in \mathbf{P}^2 : |w| < |z|\}$. For each integer $n \geq 2$ consider the map $\Phi^{(n)} : \Sigma \to \mathcal{Q}_-$ whose components are given by

$$\begin{aligned}
z_0 &= \zeta^n,\\
z_1 &= -i(z^n + z^{n-2}w^2) - i(z\overline{w} + w\overline{z})\zeta^n/(|z|^2 - |w|^2),\\
z_2 &= z^n - z^{n-2}w^2 + (z\overline{w} - w\overline{z})\zeta^n/(|z|^2 - |w|^2),\\
z_3 &= -2iz^{n-1}w - i(|z|^2 + |w|^2)\zeta^n/(|z|^2 - |w|^2).
\end{aligned}$$

Define

$$\begin{aligned}
\mathcal{A}_\nu^{(n)} &= \Big\{(z, w) \in \mathbf{C}^2 : 0 < |z|^n - |z|^{n-2}|w| < 1\Big\},\\
\mathcal{A}_\eta^{(n)} &= \Big\{(z, w) \in \mathbf{C}^2 : |z|^n - |z|^{n-2}|w|^2 > 1\Big\}
\end{aligned}$$

and regard them as subsets of the affine part of $\mathbf{P}^2$ where $\zeta = 1$. Then $\mathcal{A}_\nu^{(n)}, \mathcal{A}_\eta^{(n)} \subset \Sigma$ for all $n \geq 2$. Let $\Phi_\nu^{(n)}, \Phi_\eta^{(n)}$ be the restrictions of $\Phi^{(n)}$ to $\mathcal{A}_\nu^{(n)}, \mathcal{A}_\eta^{(n)}$ respectively. Then these maps are $n$-sheeted covering maps from $\mathcal{A}_\nu^{(n)}, \mathcal{A}_\eta^{(n)}$ onto

$$\mathcal{A}_\nu = \Big\{(z_1, z_2, z_3) \in \mathbf{C}^3 : -1 < |z_1|^2 + |z_2|^2 - |z_3|^2 < 1,\ \Im z_3 < 0\Big\} \cap \mathcal{Q}_-,$$
$$\mathcal{A}_\eta = \Big\{(z_1, z_2, z_3) \in \mathbf{C}^3 : |z_1|^2 + |z_2|^2 - |z_3|^2 > 1,\ \Im(z_2(\overline{z}_1 + \overline{z}_3)) > 0\Big\} \cap \mathcal{Q}_-$$

respectively. Equip $\mathcal{A}_\nu^{(n)}, \mathcal{A}_\eta^{(n)}$ with the pull back complex structures using $\Phi_\nu^{(n)}, \Phi_\eta^{(n)}$ respectively and call the resulting complex manifolds $M_\nu^{(n)}, M_\eta^{(n)}$ respectively. For $-1 \leq s < t \leq 1$ and $n \geq 2$ define

$$\Omega_{s,t}^{(n)} = \Big\{(z, w) \in M_\nu^{(n)} : \sqrt{(s+1)/2} < |z|^n - |z|^{n-2}|w|^2 < \sqrt{(t+1)/2}\Big\}.$$

Moreover if $\Psi_\nu : \mathcal{A}_\nu \to \mathbf{C}^2$ is given by $(z_1, z_2, z_3) \mapsto (z_1/z_3, z_2/z_3)$ then $\Psi_\nu \circ \Phi_\nu^{(n)} : \Omega_{s,t}^{(n)} \to \Omega_{s,t}$ is an $n$-sheeted cover.

On the other hand, let $\Lambda : \mathbf{C} \times \Delta \to \Sigma \cap \{\zeta = 1\}$ be the covering map given by $\Lambda(z, w) = (e^z, we^z)$ where $(z, w) \in \mathbf{C} \times \Delta$. Define

$$U_\nu = \Big\{(z, w) \in \mathbf{C}^2 : |w| < 1,\ \exp(2\Re z)(1 - |w|^2) < 1\Big\},$$
$$U_\eta = \Big\{(z, w) \in \mathbf{C}^2 : |w| < 1,\ \exp(2\Re z)(1 - |w|^2) > 1\Big\}$$

and denote by $\Lambda_\nu, \Lambda_\eta$ the restrictions of $\Lambda$ to $U_\nu, U_\eta$ respectively. Then $\Lambda_\nu : U_\nu \to M_\nu^{(2)}$ and $\Lambda_\eta : U_\eta \to M_\eta^{(2)}$ are infinite coverings. Equip $U_\nu, U_\eta$ with the pull back complex structures using $\Lambda_\nu, \Lambda_\eta$ respectively and call the resulting complex manifolds $M_\nu^{(\infty)}$ and $M_\eta^{(\infty)}$ respectively. For $-1 \leq s < t \leq 1$ define

$$\Omega_{s,t}^{(\infty)} = \Big\{(z, w) \in M_\nu^{(\infty)} : \sqrt{(s+1)/2} < \exp(2\Re z)(1 - |w|^2) < \sqrt{(t+1)/2}\Big\}$$

and note that $\Psi_\nu \circ \Phi_\nu^{(2)} \circ \Lambda_\nu : \Omega_{s,t}^{(\infty)} \to \Omega_{s,t}$ is an infinite covering.

**Proposition 4.13.** *$D$ cannot be equivalent to either $\Omega_{s,t}^{(n)}$ for $n \geq 2$ or $\Omega_{s,t}^{(\infty)}$ for all $-1 \leq s < t \leq 1$.*

*Proof.* The reasoning for $\Omega_{s,t}^{(n)}$ is subsumed by that for $\Omega_{s,t}^{(\infty)}$ and so it will suffice to focus on $\Omega_{s,t}^{(\infty)}$. Let $f : D \to \Omega_{s,t}^{(\infty)}$ be a biholomorphism. Then $\pi = \Psi_\nu \circ \Phi_\nu^{(2)} \circ \Lambda_\nu \circ f : D \to \Omega_{s,t}$ is then a holomorphic covering between domains with the standard complex structure. Note that the cluster set of $\partial D$ under $\pi$ is contained in $\partial\Omega_{s,t}$. First suppose that $-1 < s < t < 1$. Write $z_1 = x + iy, z_2 = u + iv$. The smooth hypersurfaces $\nu_t = \{|z_1|^2 + |z_2|^2 - 1 = t|z_1^2 + z_2^2 - 1|\} \setminus \{x^2 + u^2 = 1\}$ and $\nu_s = \{|z_1|^2 + |z_2|^2 - 1 = s|z_1^2 + z_2^2 - 1|\} \setminus \{x^2 + u^2 = 1\}$ are strongly pseudoconvex and strongly pseudoconcave pieces respectively of $\partial\Omega_{s,t}$. As in proposition 4.7 choose $a \in S \subset T$ such that $p_\infty \in \overline{S}$ and $S \subset \hat{D}$. Then $\pi$ extends holomorphically across $a$ and $\pi(a) \in \partial\Omega_{s,t}$. Choose neighbourhoods $U, U'$ of $a, \pi(a)$ respectively so that $\pi : U \to U'$ is well defined holomorphic and $\pi(U \cap \partial D) \subset U' \cap \partial\Omega_{s,t}$. If $\pi(a) \in \nu_t$ then it is possible to find strongly pseudoconcave points near $a$ that are mapped locally biholomorphically by $\pi$ to points on $\nu_t$ which violates the invariance of the Levi form. Likewise $\pi(a) \notin \nu_s$. The remaining possibility is that $\pi(a) \in \{(x, u) \in \mathbf{R}^2 : x^2 + u^2 = 1\} = \partial\mathcal{O}_5$. Let $Z_\pi \subset U$ be the closed analytic set defined by the vanishing of the Jacobian determinant of $\pi$. Since $\pi$ is a covering it follows that $Z_\pi \cap (U \cap D) = \emptyset$ and hence $\dim Z_\pi \leq 1$. Since $\partial D$ is of finite type near $p_\infty$ it follows that the real dimension of $Z_\pi \cap \partial D$ is at most one. Choose $p \in (U \cap S) \setminus (Z_\pi \cap \partial D)$. For reasons discussed above $\pi(p) \notin \nu_t$ or $\nu_s$. Therefore $\pi(p) \in \partial\mathcal{O}_5$ and since $p$ is arbitrary it follows that $\pi((U \cap S) \setminus (Z_\pi \cap \partial D)) \subset \partial\mathcal{O}_5$. This shows that an open piece of $S$ is mapped locally biholomorphically into $\partial\mathcal{O}_5$ and this cannot happen by dimension considerations. The only possibility that remains is that $\partial D$ is weakly pseudoconvex near $p_\infty$. By [6] it follows that $D \backsimeq \tilde{D}$ where $\tilde{D}$ is as in (4.2). Since $\tilde{D}$ is complete hyperbolic, the same must be true of $\Omega_{s,t}$. But then completeness implies that $\Omega_{s,t}$ must be pseudoconvex which cannot be true since all points on $\nu_s \subset \partial\Omega_{s,t}$ are strongly pseudoconvex.

Now suppose that $-1 = s < t < 1$. Then the boundary of $\Omega_{-1,t}$ has a strongly pseudoconvex piece $\nu_t$ and $\mathcal{O}_5 = \{(x, u) \in \mathbf{R}^2 : x^2 + u^2 < 1\}$ which is maximally totally real. For reasons discussed above

$\pi(a) \notin \nu_t$ and therefore $\pi(a) \in \overline{\mathcal{O}}_5$. As mentioned above $\dim Z_\pi \leq 1$ and hence the real dimension of $Z_\pi \cap \partial D$ is at most one. Choose $p \in (U \cap S) \setminus (Z_\pi \cap \partial D)$ and note that $\pi(p) \notin \nu_t$. Hence $\pi(p) \in \overline{\mathcal{O}}_5$. The strongly pseudoconcave points near $p$ are then mapped locally biholomorphically by $\pi$ to points on $\partial\Omega_{-1,t}$ near $\pi(p)$. However note that a neighbourhood of $\pi(p)$ on $\partial\Omega_{-1,t}$ consists entirely of either totally real points or those that are strongly pseudoconvex which again leads to a contradiction. The only possibility is that $D$ is pseudoconvex near $p_\infty$ and hence that $D \backsimeq \tilde{D}$. Thus

$$\pi : D \backsimeq \tilde{D} \to \Omega_{-1,t}$$

is also an infinite covering. Now pick $p' \in \nu_t \subset \partial\Omega_{-1,t}$ and note that there is a local holomorphic peak function at $p'$. The claim made in proposition 4.7 applies here as well and it shows the existence of $p \in \partial\tilde{D}$ such that the cluster set of $p$ under $\pi$ contains $p'$. It follows that $\pi$ extends locally biholomorphically across the strongly pseudoconvex points near $p$ and hence that $\pi$ is algebraic. In particular, for a generic $z' \in \Omega_{-1,t}$ the cardinality of $\pi^{-1}(z')$ must be finite which contradicts the fact that $\pi$ is an infinite covering map.

Similar arguments show that $D$ cannot be equivalent to $\Omega_{s,1}$ for $-1 < s < 1$. Finally it has been noted in [23] that $\Omega_{-1,1} \backsimeq \Delta^2$ whose automorphism group is six dimensional and thus $D$ cannot be equivalent to $\Omega_{-1,1}$.

□

On the other hand, for $1 \leq s < t \leq \infty$ and $n \geq 2$ define

$$\begin{aligned}
D^{(2)}_{s,t} &= \Big\{(z_1, z_2, z_3) \in \mathbf{C}^3 : s < |z_1|^2 + |z_2|^2 - |z_3|^2 < t,\ \Im(z_2(\overline{z}_1 + \overline{z}_3)) > 0\Big\} \cap \mathcal{Q}_-,\\
D^{(2n)}_{s,t} &= \Big\{(z, w) \in M^{(n)}_\eta : \sqrt{(s+1)/2} < |z|^n - |z|^{n-2}|w|^2 < \sqrt{(t+1)/2}\Big\},\\
D^{(\infty)}_{s,t} &= \Big\{(z, w) \in M^{(\infty)}_\eta : \sqrt{(s+1)/2} < \exp(2\Re z)(1 - |w|^2) < \sqrt{(t+1)/2}\Big\}.
\end{aligned}$$

The domain $D^{(2)}_{s,t}$ is a two sheeted cover of $D_{s,t}$, the covering map being $\Psi_\eta : \mathcal{A}_\eta \to \mathbf{C}^2$ given by $(z_1, z_2, z_3) \mapsto (z_2/z_1, z_3/z_1)$. Furthermore for $n \geq 2$ it can be seen that $\Psi_\eta \circ \Phi^{(n)}_\eta : D^{(2n)}_{s,t} \to D_{s,t}$ is a $2n$-sheeted covering while $\Psi_\eta \circ \Phi^{(2)}_\eta \circ \Lambda_\eta : D^{(\infty)}_{s,t} \to D_{s,t}$ is an infinite covering. The procedure for getting an $n$-sheeted cover of $D_{s,t}$ for $n$ odd has been explained in [23]. Briefly put, it is observed that there is a cyclic group of order 4 that acts freely on $D^{(4n)}_{s,t}$. The quotient space $D^{(n)}_{s,t} = \Pi^{(n)}\Big(D^{(4n)}_{s,t}\Big)$ (where $\Pi^{(n)}$ is the factorization map) is then an $n$-sheeted cover of $D_{s,t}$.

**Proposition 4.14.** *$D$ cannot be equivalent to $D^{(n)}_{s,t}$ for $n \geq 2$ or to $D^{(\infty)}_{s,t}$ for $1 \leq s < t \leq \infty$.*

*Proof.* It will suffice to show that $D$ cannot be equivalent to $D^{(\infty)}_{s,t}$ as the reasoning for $D^{(n)}_{s,t}$ is the same. So suppose that $f : D \to D^{(\infty)}_{s,t}$ is biholomorphic. Then $\pi = \Psi_\eta \circ \Phi^{(2)}_\eta \circ \Lambda_\eta \circ f : D \to D_{s,t}$ is a holomorphic covering between domains with the standard complex structure. Observe that the cluster set of $\partial D$ under $\pi$ is contained in $\partial D_{s,t}$. First suppose that $1 < s < t < \infty$. The smooth hypersurfaces $\eta_s = \{1 + |z_1|^2 - |z_2|^2 = s|1 + z_1^2 - z_2^2|,\ \Im(z_1(1 + \overline{z}_2)) > 0\}$ and $\eta_t = \{1 + |z_1|^2 - |z_2|^2 = t|1 + z_1^2 - z_2^2|,\ \Im(z_1(1+\overline{z}_2)) > 0\}$ are strongly pseudoconvex and strongly pseudoconcave pieces respectively of $\partial D_{s,t}$. The other component is $\partial D_{s,t} \cap \{\Im(z_1(1 + \overline{z}_2)) = 0\}$. As before choose $a \in S \subset T$ such that $\pi$ extends holomorphically across $a$ with $\pi(a) \in \partial D_{s,t}$. It is evident that $\pi(a) \notin \eta_s, \eta_t$. Likewise the arguments in proposition 4.13 show that $\pi(a) \notin \partial D_{s,t} \cap \{\Im(z_1(1 + \overline{z}_2)) = 0\}$ as well. Thus $\partial D$ must be weakly pseudoconvex near $p_\infty$.

Now suppose that $1 = s < t < \infty$. Again $\pi(a) \notin \eta_t$ and for similar reasons as above $\pi(a) \notin \partial D_{1,t} \cap \{\Im(z_1(1 + \overline{z}_2)) = 0\}$. Therefore $\pi(a) \in \eta_1$. To study this possibility note that $D_{1,t} \subset D_1$ and $\Psi_\eta^{-1} : D_1 \to D^{(2)}_1 \backsimeq \Delta^2$ is a proper holomorphic correspondence. The holomorphic correspondence (no longer proper) $\hat{F} = \Psi_\eta^{-1} \circ \pi : D \to \Delta^2$ extends across $a$ (since $a \in \hat{D}$) and the branches $\hat{F}_1, \hat{F}_2$ of $\hat{F}$ (there are two since $\Psi_\eta$ is generically two sheeted) satisfy $\hat{F}_1(a), \hat{F}_2(a) \in \partial\Delta^2$. The branch locus of $\hat{F}$ is of dimension at most one and since $\partial D$ is of finite type near $a$, it follows that there are strongly

pseudoconvex/pseudoconcave points near $a$ that are mapped locally biholomorphically by the branches of $\hat{F}$ into $\partial\Delta^2$. This cannot happen. Again $\partial D$ must be weakly pseudoconvex near $p_\infty$.

When $1 < s < t = \infty$, the boundary $\partial D_{1,\infty}$ contains $\eta_s$ as a strongly pseudoconvex piece, the complex curve $\mathcal{O}$ (as defined in subsection 4.5) and the remaining piece is $\partial D_{1,\infty} \cap \{\Im(z_1(1+\overline{z}_2)) = 0\} \subset \{\Im(z_1(1+\overline{z}_2)) = 0\}$. It follows from the reasoning used before that $\pi(a)$ cannot belong to any of these components. The same holds when $1 = s < t = \infty$. In any event the conclusion is that $\partial D$ must be weakly pseudoconvex near $p_\infty$. By [6] it follows that $D \backsimeq \tilde{D}$ and by the last part of proposition 4.1 it is known that $\tilde{D} \backsimeq \mathcal{D}_4 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + (\Re z_1)^{2m} < 0\}$. By assumption $D \backsimeq D^{(\infty)}_{s,t}$ which implies that $D^{(\infty)}_{s,t}$ must have a unique Levi flat orbit. However it is noted in [23] that all orbits in $D^{(\infty)}_{s,t}$ are strongly pseudoconvex hypersurfaces and this is a contradiction. □

4.7. **Miscellaneous examples:** The domain $D_s = D_{s,\infty} \cup \mathcal{O}$ as defined in section 4.5 can be regarded as a completion of $D_{s,\infty}$ by attaching the complex curve $\mathcal{O}$. Indeed, the action of $\mathrm{Aut}(D_{s,\infty})^c$ on $D_{s,\infty}$ extends to $D_s$ and $\mathcal{O}$ is an orbit for this extended action. The domain $D_{s,\infty}$ admits a finite covering by $D^{(n)}_{s,\infty} \subset M^{(n)}_\eta$ and it is natural to ask whether there is a completion of $D^{(n)}_{s,\infty}$ in the sense discussed above. The last set of examples (cf. 11(a) and 11(b)) in [23] do precisely this. Briefly put, for $1 \le s < \infty$ and $n \ge 1$ define $\mathcal{O}^{(2n)} = \{[0 : z : w] \in \mathbf{P}^2 : |w| < |z|\}$ and put $D^{(2n)}_s = D^{(2n)}_{s,\infty} \cup \mathcal{O}^{(2n)}$. It is shown that the maps $\Psi_\eta$ and $\Psi_\eta \circ \Phi^{(2n)}_\eta$ that are apriori defined only on $D^{(2)}_{s,\infty}$ and $D^{(2n)}_{s,\infty}$ respectively extend to proper holomorphic branched coverings $\Psi_\eta : D^{(2)}_s \to D_s$ and $\Psi_\eta \circ \Phi^{(2n)}_\eta : D^{(2n)}_s \to D_s$ respectively. Likewise for $n$ odd and $n \ge 3$, the map $\Pi^{(n)}$ extends to a proper holomorphic branched covering $\Pi^{(n)} : D^{(4n)}_s \to D^{(n)}_s$ and hence $\Psi_\eta \circ \Phi^{(2n)}_\eta \circ (\Pi^{(n)})^{-1} : D^{(n)}_s \to D_s$ is a proper holomorphic correspondence.

**Lemma 4.15.** *$D$ cannot be equivalent to $D^{(n)}_s$ for $n \ge 1$ and $1 \le s < \infty$.*

*Proof.* For $n$ even the existence of a biholomorphism between $D$ and $D^{(n)}_s$ would imply the existence of a proper holomorphic map from $D$ onto $D_s$. By proposition 4.12 this is not possible. On the other hand, for $n$ odd there would be a proper correspondence from $D$ onto $D_s$. The proof is the same as in proposition 4.12 once it is realised that the branch locus of the proper correspondence is of dimension one and near points of $S \subset \hat{D}$ ($S$ is the two dimensional stratum of the Levi degenerate points that clusters at $p_\infty$), the intersection of the branch locus with $\partial D$ has real dimension at most one. So there are points on $S$ near which the correspondence splits into well defined holomorphic mappings. Working with these mappings it is possible to show that $\partial D$ must be weakly pseudoconvex near $p_\infty$ and hence $D \backsimeq \tilde{D} \backsimeq \mathcal{D}_4 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + (\Re z_1)^2 < 0\}$ which has a Levi flat orbit. Thus $D^{(n)}_s$ would also have a Levi flat orbit but it has been noted in [23] that there are none in $D^{(n)}_s$. □

The other examples may be briefly recalled as follows.

$\mathfrak{D}^{(1)}_s = \Psi^{-1}_\nu(\Omega_{s,1}) \cup D^{(2)}_1 \cup \mathcal{O}^{(1)}_0,\ -1 < s < 1$

$\hat{\mathfrak{D}}^{(1)}_t = \Psi^{-1}_\nu(\Omega_1) \cup D^{(2)}_{1,t} \cup \mathcal{O}^{(1)}_0,\ 1 < t < \infty$

$\mathfrak{D}^{(1)}_{s,t} = \Psi^{-1}_\nu(\Omega_{s,1}) \cup D^{(2)}_{1,t} \cup \mathcal{O}^{(1)}_0,\ -1 \le s < 1 < t \le \infty$, where $s = -1$ and $t = \infty$ do not hold simultaneously

$\mathfrak{D}^{(n)}_s = \Omega^{(n)}_{s,1} \cup D^{(2n)}_1 \cup \mathcal{O}^{(n)}_0,\ -1 < s < 1$

$\mathfrak{D}^{(n)}_{s,t} = \Omega^{(n)}_{s,1} \cup D^{(2n)}_{1,t} \cup \mathcal{O}^{(n)}_0,\ -1 \le s < 1 < t \le \infty$, where $s = -1$ and $t = \infty$ do not hold simultaneously

$\mathfrak{D}^{(\infty)}_s = \Omega^{(\infty)}_{s,1} \cup D^{(\infty)}_{1,\infty} \cup \mathcal{O}^{(\infty)}_0,\ -1 < s < 1$

$\mathfrak{D}^{(\infty)}_{s,t} = \Omega^{(\infty)}_{s,1} \cup D^{(\infty)}_{1,t} \cup \mathcal{O}^{(\infty)}_0,\ -1 < s < 1 < t \le \infty$, where $s = -1$ and $t = \infty$ do not hold simultaneously

where $\mathcal{O}^{(1)}_0$ is a Levi flat orbit and $\mathcal{O}^{(n)}_0, \mathcal{O}^{(\infty)}_0$ are its $n$-sheeted and infinite covering respectively. The details of this construction are given in [23], but the relevant point here is that all have Levi flat orbits.

**Proposition 4.16.** *$D$ cannot be equivalent to any of the domains listed above.*

*Proof.* It will suffice to show that $D$ cannot be equivalent to $\mathfrak{D}^{(1)}_s$ for similar arguments can be applied in all the other cases. If $D \backsimeq \mathfrak{D}^{(1)}_s$ for $-1 < s < 1$ then $D$ must have a Levi flat orbit and by proposition

4.1 it follows that
$$D \backsimeq \mathcal{D}_4 \backsimeq R_{1/2m,-1,1} = \Big\{(z_1,z_2) \in \mathbf{C}^2 : -(\Re z_1)^{1/2m} < \Re z_2 < (\Re z_1)^{1/2m},\ \Re z_1 > 0\Big\}.$$
The orbits in $R_{1/2m,-1,1}$, apart from a unique Levi flat orbit namely $\mathcal{O}_1$, are strongly pseudoconvex hypersurfaces of the form $O_\alpha^{R_{1/2m}} = \{\Re z_2 = \alpha(\Re z_1)^{1/2m};\ \Re z_1 > 0\}$ for $-1 < \alpha < 1$. On the other hand the orbits of points in $\Psi_\nu^{-1}(\Omega_{s,1}) \subset \mathfrak{D}_s^{(1)}$ are equivalent to $\nu_\alpha = \{|z_1|^2 + |z_2|^2 - 1 = \alpha|z_1^2 + z_2^2 - 1|\} \setminus \{(x,u) \in \mathbf{R}^2 : x^2 + u^2 = 1\}$ for $s < \alpha < 1$. The hypersurfaces $O_\alpha^{R_{1/2m}}$ and $\nu_\alpha$ are not CR-equivalent (cf. [25]) and this is a contradiction. ☐

## 5. Model domains when Aut($D$) is four dimensional

It is shown in [24] (compare with the result in [32]) that there are exactly 7 isomorphism classes of Kobayashi hyperbolic manifolds of dimension two whose automorphism group has dimension four. These are listed below along with some properties that are relevant to this discussion and the idea once again will be to show that $D \backsimeq \mathcal{D}_5 = \{(z_1,z_2) \in \mathbf{C}^2 : 2\Re z_2 + |z_1|^{2m} < 1\}$, where $m \geq 2$ is an integer, by eliminating all other possibilities from this list.

• Let $S_r = \{z \in \mathbf{C}^2 : r < |z| < 1\}$ where $0 \leq r < 1$ be a spherical shell. The automorphism group of this domain is the unitary group $U_2$. Evidently $D$ cannot be equivalent to $S_r$ since Aut($D$) is non-compact by assumption. Quotients of $S_r$ can also be obtained by realising $\mathbf{Z}_m, m \in \mathbf{N}$ as a subgroup of scalar matrices in $U_n$ and considering $S_r/\mathbf{Z}_m$. This has fundamental group $\mathbf{Z}_m$. Clearly $D$ being simply connected cannot be equivalent to $S_r/\mathbf{Z}_m$.

• Define $\mathcal{E}_{r,\theta} = \{(z_1,z_2) \in \mathbf{C}^2 : |z_1| < 1,\ r(1-|z_1|^2)^\theta < |z_2| < (1-|z_1|^2)^\theta\}$ where $\theta \geq 0, 0 \leq r < 1$ or $\theta < 0, r = 0$.

(i) When $\theta > 0$ and $0 < r < 1$ the boundary of $\mathcal{E}_{r,\theta}$ consists of the spherical hypersurfaces $\{|z_1| < 1,\ |z_1|^2 + (|z_2|/r)^{1/\theta} = 1\}$ and $\{|z_1| < 1,\ |z_1|^2 + |z_2|^{1/\theta} = 1\}$ and the circle $\{|z_1| = 1,\ z_2 = 0\}$. Note that $z_2 \neq 0$ on either of the hypersurfaces as otherwise $|z_1| = 1$.

(ii) When $\theta > 0$ and $r = 0$ the domain is $\mathcal{E}_{0,\theta} = \{(z_1,z_2) \in \mathbf{C}^2 : |z_1| < 1,\ 0 < |z_2| < (1-|z_1|^2)^\theta\}$. The boundary of this domain consists of the spherical hypersurface $\{|z_1| < 1,\ |z_1|^2 + |z_2|^{1/\theta} = 1\}$ and the closed unit disc $\{|z_1| \leq 1,\ z_2 = 0\}$.

(iii) When $\theta = 0$ and $0 \leq r < 1$ the domain is $\mathcal{E}_{r,0} = \{(z_1,z_2) \in \mathbf{C}^2 : |z_1| < 1,\ r < |z_2| < 1\}$ which is not simply connected. The same holds when $\theta < 0$ and $r = 0$ in which case the domain is $\mathcal{E}_{0,\theta} = \{(z_1,z_2) \in \mathbf{C}^2 : |z_1| < 1,\ 0 < |z_2| < (1-|z_1|^2)^\theta\}$. Hence $D$ cannot be equivalent to either $\mathcal{E}_{r,0}$ or $\mathcal{E}_{0,\theta}$.

• Define $\Omega_{r,\theta} = \{(z_1,z_2) \in \mathbf{C}^2 : |z_1| < 1,\ r(1-|z_1|^2)^\theta < \exp(\Re z_2) < (1-|z_1|^2)^\theta\}$ where $\theta = 1, 0 \leq r < 1$ or $\theta = -1, r = 0$.

(i) When $\theta = 1, 0 \leq r < 1$ the domain is $\Omega_{r,1} = \{(z_1,z_2) \in \mathbf{C}^2 : |z_1| < 1,\ r(1-|z_1|^2) < \exp(\Re z_2) < 1 - |z_1|^2\}$ and its boundary has two components $\{|z_1| < 1,\ r(1-|z_1|^2) = \exp(\Re z_2)\}$ and $\{|z_1| < 1,\ 1 - |z_1|^2 = \exp(\Re z_2)\}$ both of which are spherical.

(ii) When $\theta = -1, r = 0$ the domain is $\Omega_{0,-1} = \{(z_1,z_2) \in \mathbf{C}^2 : |z_1| < 1,\ 0 < \exp(\Re z_2) < 1/(1 - |z_1|^2)^{-1}\}$. The map $(z_1,z_2) \mapsto (z_1,-z_2)$ transforms $\Omega_{0,-1}$ biholomorphically onto $\{(z_1,z_2) \in \mathbf{C}^2 : |z_1| < 1,\ 1 - |z_1|^2 < \exp(\Re z_2)\}$ whose boundary $\{|z_1| < 1,\ 1 = |z_1|^2 + \exp(\Re z_2)\}$ is spherical.

• Define $D_{r,\theta} = \{(z_1,z_2) \in \mathbf{C}^2 : r\exp(\theta|z_1|^2) < |z_2| < \exp(\theta|z_1|^2)\}$ where $\theta = 1, 0 < r < 1$ or $\theta = -1, r = 0$.

(i) When $\theta = 1$ and $0 < r < 1$ the domain is $D_{r,1} = \{(z_1,z_2) \in \mathbf{C}^2 : r\exp(|z_1|^2) < |z_2| < \exp(|z_1|^2)\}$ and its boundary has two components $\{|z_2| = \exp(r|z_1|^2)\}$ and $\{|z_2| = \exp(r|z_1|^2)\}$ both of which are spherical.

(ii) When $\theta = -1$ and $r = 0$ the domain is $D_{0,-1} = \{(z_1,z_2) \in \mathbf{C}^2 : 0 < |z_2| < \exp(-|z_1|^2)\}$ which is mapped biholomorphically by $(z_1,z_2) \mapsto (z_1, 1/z_2)$ onto $\{(z_1,z_2) \in \mathbf{C}^2 : \exp(|z_1|^2) < |z_2|\}$. Note that the boundary of this is again spherical.

• Define $\mathfrak{S} = \{(z_1, z_2) \in \mathbf{C}^2 : -1 + |z_1|^2 < \Re z_2 < |z_1|^2\}$. It can be seen that both boundary components of this domain are spherical.

• Define $\mathcal{E}_\theta = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1| < 1,\ |z_2| < (1 - |z_1|^2)^\theta\}$ where $\theta < 0$. The boundary of $\mathcal{E}_\theta$ consists of $L = \{|z_1| = 1\} \times \mathbf{C}_{z_2}$ which is Levi flat and $S = \{|z_1| < 1,\ |z_2| = (1 - |z_1|^2)^\theta\}$ where $\theta < 0$. Choose $p = (p_1, p_2) \in S$. Note that $p_2 \neq 0$ and hence the mapping $(z_1, z_2) \mapsto (z_1, 1/z_2)$ is well defined near $p$ and maps a germ of $S$ near $p$ biholomorphically to a germ of $\{|z_1| < 1,\ (1 - |z_1|^2)^{-\theta} = |z_2|\}$ which is seen to be spherical. Moreover $S$ viewed from within $\mathcal{E}_\theta$ is a strongly pseudoconcave point as a straightforward computation shows.

• Define $E_\theta = \{(z_1, z_2) \in \mathbf{C}^2 : |z_1|^2 + |z_2|^\theta < 1\}$ where $\theta > 0, \theta \neq 2$.

**Proposition 5.1.** *$D$ cannot be equivalent to any of $\mathcal{E}_{r,\theta}$ (where $\theta > 0, 0 \leq r < 1$), $\Omega_{r,\theta}$ (where $\theta = 1, 0 \leq r < 1$ or $\theta = -1, r = 0$), $D_{r,\theta}$ (where $\theta = 1, 0 < r < 1$ or $\theta = -1, r = 0$), $\mathfrak{S}$ or to $\mathcal{E}_\theta$ (where $\theta < 0$).*

*Proof.* The domains listed above have a common feature namely that a large piece of their boundary (if not all) is spherical. The argument is essentially the same for all the domains and it will therefore suffice to illustrate the reasoning in two cases, namely $\mathcal{E}_{0,\theta}$ and $\mathcal{E}_\theta$. Suppose that $f : D \to \mathcal{E}_{0,\theta}$ is a biholomorphism. Note that $\psi(z) = \log |z_2|$ is plurisubharmonic everywhere and its negative infinity locus contains $\overline{\Delta} = \{|z_1| \leq 1,\ z_2 = 0\} \subset \partial\mathcal{E}_{0,\theta}$. Fix an open neighbourhood $U$ of $p_\infty \in \partial D$. Let $\Gamma \subset U \cap \partial D$ be the set of those points whose cluster set is entirely contained in $\overline{\Delta}$. If $\Gamma$ contains a relatively open subset of $\partial D$ then the uniqueness theorem shows that $\psi \circ f$, which is plurisubharmonic on $D$, must satisfy $\psi \circ f \equiv -\infty$ on $D$. This is a contradiction. Therefore $\Gamma \subset U \cap \partial D$ is nowhere dense and hence it is possible to choose a strongly pseudoconcave point $p \in (U \cap \partial D) \setminus \Gamma$. Then $f$ extends to a neighbourhood of $p$ and $f(p) \in \Sigma = \{|z_1| < 1,\ |z_1|^2 + |z_2|^{1/\theta} = 1\}$. Let $g$ be a local biholomorphism defined in an open neighbourhood of $f(p)$ that maps the germ of $\Sigma$ near $f(p)$ to $\partial\mathbf{B}^2$. Then $g \circ f$ is a biholomorphic germ at $p$ that maps the germ of $\partial D$ at $p$ into $\partial\mathbf{B}^2$. By [40] this germ of a mapping can be analytically continued along all paths in $U \cap \partial D$ that start at $p$. In particular there is an open neighbourhood $\tilde{U}$ of $p_\infty$, a holomorphic mapping $\tilde{f} : \tilde{U} \to \mathbf{C}^2$ such that $\tilde{f}(\tilde{U} \cap \partial D) \subset \partial\mathbf{B}^2$. This shows that $\partial D$ must be weakly pseudoconvex near $p_\infty$ and moreover $p_\infty$ must a weakly spherical point by [11]. By [6] it follows that $D \backsimeq \mathcal{D}_5 = \{(z_1, z_2) \in \mathbf{C}^2 : 2\Re z_2 + |z_1|^{2m} < 0\} \backsimeq E_{2m}$ and therefore $E_{2m} \backsimeq \mathcal{E}_{0,\theta}$ which is a contradiction.

On the other hand suppose that $f : D \to \mathcal{E}_\theta$ is biholomorphic. Choose $p \in \partial D$ a strongly pseudoconcave point near $p_\infty$. Then $f$ extends to a neighbourhood of $p$ and $f(p) \in \partial\mathcal{E}_\theta$. By shifting $p$ if necessary it can be assumed that $f$ is locally biholomorphic near $p$. Note that $f(p) \notin L$ as $\partial D$ is assumed to be of finite type near $p_\infty$. If $f(p) \in S$ then we may compose $f$ with $g$ as above and get a germ of a holomorphic mapping from a neighbourhood of $p \in \partial D$ into $\partial\mathbf{B}^2$. As above this can be analytically continued to get a holomorphic mapping defined in an open neighbourhood of $p_\infty$ that takes $\partial D$ into $\partial\mathbf{B}^2$. This leads to the conclusion that $E_{2m} \backsimeq \mathcal{E}_\theta$ which is false. The only other possibility is that $\partial D$ is weakly pseudoconvex near $p_\infty$. By [6] it follows that $D \backsimeq \tilde{D}$ where $\tilde{D}$ is as in (4.2). Therefore $\mathcal{E}_\theta$ must be pseudoconvex as well. However all points on $S \subset \partial\mathcal{E}_\theta$ are pseudoconcave points. □

The only possibility is that $D \backsimeq E_\theta$ for some $\theta > 0, \theta \neq 2$. Clearly $E_\theta$ is pseudoconvex and so must $D$ be. Note that $\partial E_\theta$ is spherical except along the circle $\{(e^{i\alpha}, 0)\}$. Using the plurisubharmonic function $\psi(z)$ in exactly the same way as in proposition 5.1, it can be seen that there are points $p \in \partial D$ near $p_\infty$ such that the cluster set of $p$ under the biholomorphism $f : D \to E_\theta$ contains points of $\partial E_\theta \setminus \{(e^{i\alpha}, 0)\}$. Then $f$ will extend holomorphically to an open neighbourhood of $p$ and $f(p) \in \partial E_\theta \setminus \{(e^{i\alpha}, 0)\}$. As above we may compose $f$ with $g$ to get a germ of a holomorphic mapping from a neighbourhood of $p \in \partial D$ into $\partial\mathbf{B}^2$. By continuation this will give rise to a map from a neighbourhood of $p_\infty$ on $\partial D$ into $\partial\mathbf{B}^2$. It follows from [11] that $p_\infty$ must be weakly spherical and hence [6] shows that $D \backsimeq E_{2m}$ where $m \geq 2$ is an integer.

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Department of Mathematics, Indian Institute of Science, Bangalore 560 012, India
*E-mail address*: `kverma@math.iisc.ernet.in`